Université de Rennes

M2 Maths Fonda

Spectral Theory

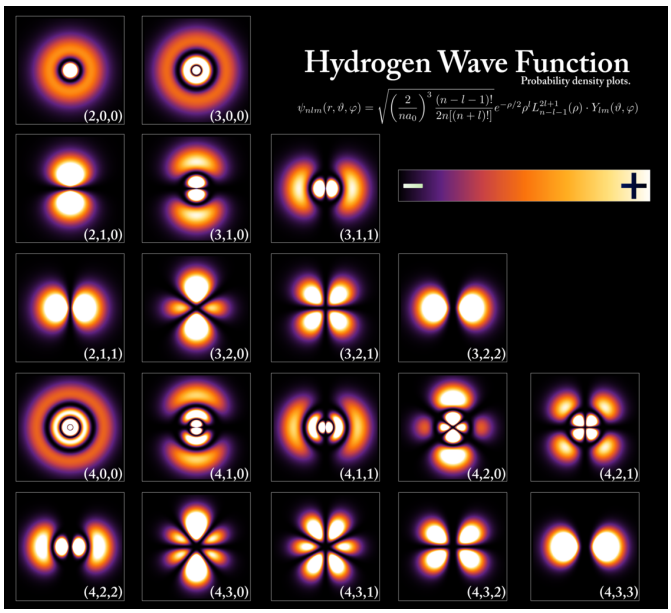


Figure 1: Wavefunctions of the electron in a hydrogen atom at different energy levels. Energy levels of a quantum system are the eigenvalues of its Hamiltonian, and wavefunctions are associated eigenvectors.

SABIN Julien

2024-2025

Notes de cours : MIANNAY Matteo

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Chapter 1

Spectral Theory of bounded operators

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1.1 Definitions, examples

In this class, $\mathcal{H}$ is a $\mathbb{C}$-Hilbert space. Its inner product is denoted by $\langle \cdot \rangle$, and we take the following conventions : $\forall u, v \in \mathcal{H}, \lambda \in \mathbb{C}, \langle \lambda u \cdot v \rangle = \bar{\lambda} \langle u \cdot v \rangle$. Usual examples are given by $\mathbb{C}^d, \ell^2(\mathbb{N}), L^2(\Omega)$....

Definition 1 (Bounded operator)**.** *A bounded operator on $\mathcal{H}$ is a linear map $\mathcal{H} \longrightarrow \mathcal{H}$ which is continuous. We denote the set of bounded operators on $\mathcal{H}$ by $\mathcal{B}(\mathcal{H})$ or $\mathcal{L}_c(\mathcal{H})$.*

Property 2. *$\mathcal{B}(\mathcal{H})$ is a Hilbert space for the norm $|||T||| = \sup_{||u||=1} ||Tu||$*

Definition 3 (Resolvant set, spectrum)**.** *The resolvent set of T, denoted by $\rho(T)$, is the set of $\lambda \in \mathbb{C}$ such hat $T - \lambda I_d$ is invertible in $\mathcal{B}(\mathcal{H})$. The spectrum of T, denoted by $\sigma(T)$ or Spec(T) is the set $\sigma(T) = \mathbb{C} - \rho(T)$.*

Remark 4. *The sentence "is invertible in $\mathcal{B}(\mathcal{H})$" implies that $T - \lambda I_d$ is invertible and has a continuous inverse. Or, since $\mathcal{H}$ is complete, the Banach isomorphism theorem can be applied and tells us that if $T - \lambda I_d$ is invertible in $\mathcal{L}(\mathcal{H})$, it is invertible in $\mathcal{B}(\mathcal{H})$.*

Remark 5. *In finite dimension, $\lambda \in \sigma(T) \iff det(T - \lambda I_d) = 0 \iff \lambda$ is an eigenvalue of T. This does not work if the dimension is not finite, as we will see.*

1.2 First computations

Example 6. *Let $\mathcal{H} = \ell^2(\mathbb{N})$, T the left shift operator : $T(u_0, ...) = (u_1, u_2, ...)$. We will show that $\sigma(T)$ is the closed unit disk.*

First, we'll check for eigenvalues : $Tu = \lambda u \iff \forall n \in \mathbb{N}, u_{n+1} = \lambda u_n$, which gives us $u_n = \lambda^n u_0$. Such a sequence is non-zero and in $\ell^2(\mathbb{N})$ if and only if we have $u_0 \neq 0$ and $|\lambda| < 1$. This shows us that the set of eigenvalues for T is the unit open disk (and therefore it is not equal to the spectrum of T). So far, we've shown that $\{\lambda \in \mathbb{C}, |\lambda| < 1\} \subset \sigma(T)$. Let's show that $\mathbb{U} \subset \sigma(T)$. Let $\lambda \in \mathbb{U}$, we've just shown that $ker(T - \lambda I_d) = 0$. We are going to define a sequence of "almost eigenvectors". Indeed, since an eigenvector has to be of the form $u_n = \lambda^n u_0$, we will try to approach an eigenvector by this sequence of sequences : $u_n^k = (1, \lambda, \lambda^2, ..., \lambda^k, 0, 0...)$. An easy calculation show that $(T - \lambda I_d)(u^{(k)}) = (0, , , 0, \lambda^{k+1}, 0, ..)$, the λ_{k+1} being in k-th position. Since $\lambda \in \mathbb{U}, ||(T - \lambda I_d)u^{(k)}|| = 1$, and if $v^{(k)} = \dfrac{u^{(k)}}{||u^{(k)}||} = \dfrac{u^{(k)}}{\sqrt{k+1}}$, we have $||(T - \lambda I_d)v^{(k)}|| = \dfrac{1}{\sqrt{k+1}} \longrightarrow 0$, where $||v^{(k)}|| = 1$.

If $T - \lambda I_d$ was invertible, since $(T - \lambda I_d)v^{(k)} \longrightarrow 0$, we would have, when composing with $(T - \lambda I_d)^{-1}$, $v^{(k)} \longrightarrow 0$, which contradicts $||v^{(k)}|| = 1$.

To conclude, we have to show that $\sigma(T)$ is included in the unit disk. To do that, we show that if $|\lambda| > 1$, then $\lambda \in \rho(T)$. We already know that $T - \lambda I_d$ is injective (the first part of our calculations), we thus have to show that $T - \lambda I_d$ is surjective. Let $v \in \mathcal{H}$. Some calculations invite us to define $u_n = -\sum_{k=0}^{\infty} \dfrac{v_{n+k}}{\lambda^{k+1}}$. This is well defined, and $(T - \lambda I_d)u = v$.

Remark 7. *We will have some topological results which will help us avoiding some calculations later on.*

Example 8. *Let $\mathcal{H} = \ell^2(\mathbb{Z})$, $T : \mathcal{H} \longrightarrow \mathcal{H}, u \mapsto (u_{n+1})$. Show that $T \in \mathcal{B}(\mathcal{H})$, and determine $\sigma(T)$.*

T is bounded because for all $u \in \mathcal{H}$, $||Tu|| = ||u||$. T has no eigenvectors, because an eigenvector would be of the form $u_n = u_0\lambda^n$, and λ^n is never summable over $\mathbb{Z}$. The exact same calculations as earlier show that the spectrum is actually $\mathbb{U}$. The only difference is that as an almost eigenvectors, you have to add terms on the left : $(0, ..., \lambda^{-k}, ..., 1, \lambda, \lambda^2, ...\lambda^k, 0, ...0..)$

Example 9. *$\mathcal{H} = L^2(]0, 1[, \mathbb{C})$, and $\forall f \in \mathcal{H}, Af(x) = xf(x) a.e.$ Then A is bounded, and $\sigma(A) = [0, 1]$. A is bounded :*

$$||A(f)||^2 = \int_0^1 |Af(x)|^2 dx = \int_0^1 |x^2| |f(x)|^2 dx \leq ||f||^2.$$

If we try and find eigenvalues : $Af = \lambda f$ tells us that $xf(x) = \lambda f(x) a.e$, so $(x - \lambda)f(x) = 0\, a.e$, and $f = 0 \in \mathcal{H}$. A has no eigenvalues. Let $\lambda \in [0, 1]$. We show that $\lambda \in \sigma(A)$. If we define $f_n(x) = \chi_{|x-\lambda| \leq \frac{1}{n}}$, then $||f_n||^2 = \dfrac{C}{n}$ and $||(A - \lambda I_d)f_n||^2 = \dfrac{C'}{n^3}$ (just write it down). Then, $\dfrac{||(A - \lambda I_d)f_n||}{||f_n||} \longrightarrow 0$, which shows that $\lambda \in \sigma(A)$ (same reasons as for the first example). (We

chose f_n so that it is concentrated around λ) Now, let's show that show that $\sigma(A) \subset [0,1]$. To do that, let $\lambda \in \mathbb{C} - [0,1]$. We know that $A - \lambda I_d$ is injective, we have to show that it is surjective. For that, let $g \in \mathcal{H}$, we define $f : x \mapsto \frac{g(x)}{x-\lambda}$. This is well defined because $\lambda \notin [0,1]$ so $x \mapsto \frac{1}{x-\lambda}$ is bounded. Then $(A - \lambda I_d)f = g$.

Remark 10. *If $m \in \mathcal{C}([0;1], \mathbb{C}), A : \mathcal{H} \longrightarrow \mathcal{H}, f \mapsto (x \mapsto m(x)f(x))$, then $\sigma(A) = m([0,1])$. A is called a multiplication operator. These generalize diagonal matrixes.*

1.3 General properties of the spectrum

Theorem 11. *Let $A \in \mathcal{B}(\mathcal{H})$.*

1. *$\rho(A)$ is open in $\mathbb{C}$.*
2. *$\sigma(A) \subset \overline{B(0, ||A||)}$*
3. *$\sigma(A)$ is non-empty.*
4. *$\sigma(A)$ is a non-empty, compact set.*

Proof.

1. Let $\lambda_0 \in \rho(A)$. We have : $A - \lambda I_d = A - \lambda_0 I_d + (\lambda_0 - \lambda) I_d = (A - \lambda_0 I_d)(I_d - (\lambda - \lambda_0)(A - \lambda_0 I_d)^{-1})$. The second term of the product is invertible when $(\lambda - \lambda_0) \leq \frac{1}{||(A - \lambda_0 I_d)^{-1}||}$ (Neumann criteria). This concludes.
2. If $|\lambda| > ||A||$, then : $A - \lambda I_d = -\lambda(I_d - \frac{A}{\lambda})$, which is invertible ($\lambda \neq 0$ and Neumann criteria).
3. If $\sigma(A) = \emptyset$, then $\rho(A) = \mathbb{C}$. Let $u, v \in \mathcal{H}$, $\lambda \in \mathbb{C} \mapsto \langle u \cdot (A - \lambda I_d)^{-1} v \rangle$ is analytic over $\mathbb{C}$, but goes to 0 as λ goes to infinity (write the inverse of $(A - \lambda I_d)$ as a power serie). Therefore, $\langle u \cdot (A - \lambda I_d)^{-1} v \rangle = 0$ for all $u, v \in \mathcal{H}, \lambda \in \mathbb{C}$, and $(A - \lambda I_d)^{-1} = 0$: absurd.
4. The above.

□

Chapter 2

Spectral theorem for bounded operators

We are going to try and extend the spectral theorem to operators.

2.1 Adjoint

Definition 12 (Adjoint)**.** *The adjoint of $A \in \mathcal{B}(\mathcal{H})$, denoted by A^*, is the unique bounded operator such that for all $u, v \in \mathcal{H}$, $\langle u \cdot Av \rangle = \langle A^* u \cdot v \rangle$.*

Remark 13. *Existence and unicity of such an operator is a consequence of Riesz theorem.*

Example 14.

1. *The left-shift operator adjoint is the right-shift operator.*
2. *$\mathcal{H} = L^2(]0,1[)$, $Tf : f \mapsto (x \mapsto \int_0^y f(t)dt)$. Show that it is well defined and compute T^*.*

Definition 15. *If $B^* = B$, we say that B is self adjoint.*

Example 16. *If (X, μ) is a measure space, $m \in L^\infty(X, \mathbb{R})$, the multpilication operator $m(L^2(X, \mu, \mathbb{C}) \longrightarrow (L^2(X, \mu, \mathbb{C}))$, $f \mapsto mf$ is a self adjoint operator.*

Property 17.

1. $(A^*)^* = A$.
2. $||A|| = ||A^*||$.
3. $(A + B)^* = A^* + B^*$.
4. $(\lambda A)^* = \overline{\lambda} A^*$.
5. $(AB)^* = B^* A^*$.
6. *If A is invertible,* $(A^*)^{-1} = (A^{-1})^*$
7. $\sigma(A^*) = \overline{\sigma(A)}$.
8. *If $B = B^*$ (self-adjoint), then* $||B|| = \sup_{||u||=1} |\langle Bu \cdot u \rangle|$.
9. $||A^* A|| = ||A||^2$.

Proof. Only the two latters are problematic. For 8 : $||B|| = \sup_{||u||=1} ||Bu|| = \sup_{||u||=||v||=1} |\langle v \cdot Bu \rangle|$.
We have to prove : $\sup_{||u||=||v||=1} |\langle v \cdot Bu \rangle| = \sup_{||u||=1} |\langle u \cdot Bu \rangle|$.

Since B is self-adjoint, $\langle (u+v)\cdot B(u+v)\rangle = \langle u\cdot Bu\rangle + \langle v\cdot Bv\rangle + 2Re(\langle v\cdot Bu\rangle)$ and $\langle (u-v)\cdot B(u-v)\rangle = \langle u\cdot Bu\rangle + \langle v\cdot Bv\rangle - 2Re(\langle v\cdot Bu\rangle)$, which gives us the polarization identity :

$$4Re(\langle v\cdot Bu\rangle) = \langle (u+v)\cdot B(u+v)\rangle - \langle (u-v)\cdot B(u-v)\rangle.$$

We deduce :

$$4|Re(\langle v\cdot Bu\rangle)| \leq |\langle u+v\cdot B(u+v)\rangle| + |\langle u-v\cdot B(u-v)\rangle| \leq S(||u+v||^2+||u-v||^2) = 2S(||u||^2+||v||^2),$$

where $S = \sup_{||u||=1} ||Bu||$. Applying this with $u = e^{-i\theta}u$, where θ is chosen to make the scalar product real, we get :

$$4|\langle v, Bu\rangle| \leq 2S(||u||^2 + ||v||^2),$$

which gives us :

$$4 \sup_{||u||=||v||=1} (\langle v\cdot Bu) \leq 4S.$$

This proves $\sup_{||u||=||v||=1} Bu = \sup_{||u||=1} \langle u\cdot Bu\rangle$.

Thanks to this, we prove 9 : $||A^*A|| = \sup_{||u||=1} \langle A^*Au\cdot u\rangle = \sup_{||u||=1} ||Au|| = ||A||$. □

Remark 18. *The identity $||B|| = \sup_{||u||=1} |\langle Bu\cdot u\rangle|$ is wrong in general. For instance, if B is antisymetric ($B^* = -B$), we have $\langle Bu\cdot u\rangle = 0$ for all $u \in \mathcal{H}$.*

Property 19. *If B is self-adjoint, then $\sigma(B) \subset \mathbb{R}$.*

Proof. Let $\lambda \in \sigma(B)$. We suppose that $Im(\lambda) \neq 0$. Let $u \in \mathcal{H}$.
$||(A-\lambda I_d)u||^2 = \langle u\cdot(A-\overline{\lambda}I_d)(A-\lambda I_d)u\rangle = \langle u\cdot((A-Re(\lambda)I_d)+iI_m\lambda I_d)((A-Re(\lambda I_d))-iIm(\lambda)I_d)u\rangle = ||(A-Re(\lambda)I_d)u||^2 + (Im(\lambda))^2||u||^2$. In particuliar :

$$||(A-\lambda I_d)u|| \geq |Im(\lambda)||u||.$$

This proves that : $(A-\lambda I_d)$ is injective, and of closed range (just write it down, and it works...).

We just have to show that $Ran(A-\lambda I_d)$ is dense. For that, we use Riesz criteria for density, and show that $Ran(A-\lambda I_d)^\perp = \{0\}$. Let $v \in Ran(A-\lambda I_d)^\perp$. For all $u \in \mathcal{H}$, $\langle v\cdot(A-\lambda I_d)u\rangle = 0 = \langle (A-\overline{\lambda}I_d)v\cdot u\rangle)$. This proves that $(A-\overline{\lambda}I_d)v = 0$. But since $(A-\overline{\lambda}I_d)$ is injective (thanks to the previous argument, applied with $\overline{\lambda}$) we get $v = 0$, which concludes : $Ran(A-\lambda I_d) = \mathcal{H}$ since it's closed and dense. □

2.2 Functional Calculus

We want to prove the following :

Theorem 20 (Spectral theorem)**.** *If A is self-adjoint, there exists a mesured set (X, μ) and a unitary operator $U : L^2(X, \mu) \longrightarrow \mathcal{H}$ such that U^*AU is a multiplication operator on H.*

To do that, we have to introduce functional calculus.

Property 21. *Let $P \in \mathbb{C}[X]$. Then $\sigma(P(A)) = P(\sigma(A))$.*

Proof. Let $\lambda \in \sigma(A)$. Let $Q \in \mathbb{C}[X]$ such that $P(X) - P(\lambda) = (X - \lambda)Q(X)$. If $P(\lambda) \in \rho(P(A))$, let B be such that $B(P(A) - P(\lambda)I_d) = I_d$. Then $BQ(A)(A - \lambda I_d) = I_d$ and $A - \lambda I_d$ is invertible : absurd. This tells us that $P(\lambda) \in \sigma(P(A))$.

Now, let $\mu \in \sigma(P(A))$. If P is constant, $P(A) = cI_d$ and $\sigma(P(A)) = \{c\} = P(\sigma(A))$. If P is non-constant, then $P(X) - \mu = a_0(X - \lambda_1)....(X - \lambda_n)$. This gives us : $P(A) - \mu I_d = a_0(A - \lambda_1 I_d).....(A - \lambda_n)I_d$. One of the terms of the product has to be non invertible since the term of the left is non invertible. So one of the λ_j is in $\sigma(A)$ and $P(\lambda_j) = \mu$, and we're done. □

Property 22. *Let A be a self adjoint bounded operator. Then $||A|| = \lim_n ||A^n||^{\frac{1}{n}} = \sup_{\lambda \in \sigma(A)} |\lambda|$.*

Proof. **For the first equality** : $a_n = \log ||A^n||$ is sub-additive. Therefore, $\frac{a_n}{n}$ converges in $[-\infty, +\infty[$ (Fekete Lemma), which shows that $||A^n||^{\frac{1}{n}}$ converges (take the exponential). Moreover, since $A = A^*$, $||A||^2 = ||A^2||$, and for all $k \in \mathbb{N}, ||A^{2^k}||^{\frac{1}{2^k}}$. Taking the limit, we have : $||A|| = \lim_n ||A^{2^n}||^{\frac{1}{2^n}} = \lim_n ||A^n||^{\frac{1}{n}}$.

Now we prove the **second equality** : we already know that $r := \sup_{\lambda \in \sigma(A)} |\lambda| \leq ||A||$. Since $\lambda \mapsto (A - \lambda I_d)^{-1}$ is analytic on $\{\lambda \in \mathbb{C}, |\lambda| > r\}$, $\mu \mapsto (I_d - \mu A)^{-1}$ is analytic on $\left\{\mu, |\mu| < \frac{1}{r}\right\}$

Its power series at 0 is $\sum \mu^n A^n$, whose radius of convergence R checks the following : $R \geq \frac{1}{r}$ and $\frac{1}{R} = \limsup ||A^n||^{\frac{1}{n}} = ||A||$. This gives us $||A|| \leq r$, which concludes. □

Property 23. *Let A be self-adjoint, $P \in \mathbb{C}[X]$. Then, $||P(A)|| = \sup_{\lambda \in \sigma(A)} |P(\lambda)|$.*

Remark 24. *Since $P \in \mathbb{C}[X]$, $P(A)$ is not self-adjoint a priori : this is a stronger result than the previous one.*

Proof. $||P(A)||^2 = ||P(A)^*P(A)|| = ||\overline{P}(A)P(A)|| = ||P\overline{P}(A)||$. $\overline{P}P(A)$ is self-adjoint because $\overline{P}P \in \mathbb{R}[X]$, which gives us : $||P(A)||^2 = \sup_{\mu \in \sigma(\overline{P}P(A))} |\mu| = \sup_{\lambda \in \sigma(A)} |\overline{P}P(\lambda)| = \sup_{\lambda \in \sigma(A)} |P(\lambda)|^2$, which concludes. □

Theorem 25 (Continuous functional calculus)**.** *Let A be a self-adjoint operator. Then there exists a unique map :*

$$\varphi : \mathcal{C}(\sigma(A), \mathbb{C}) \longrightarrow \mathcal{B}(\mathcal{H})$$

which satisfies :

1. $\varphi(f+g) = \varphi(f) + \varphi(g)$
2. $\varphi(fg) = \varphi(f)\varphi(g)$
3. $\varphi(f)^* = \varphi(\overline{f})$
4. $\varphi(1) = I_d$
5. $\varphi(i_d) = A$
6. φ *is continuous.*

This theorem extends calculations on A : we denote $\varphi(f)$ by $f(A)$. This makes sense because $\varphi(P) = P(A), \forall P \in \mathbb{C}[X]$.

Property 26. *Furthermore,* φ *verifies :*

1. *If* $A\phi = \lambda\phi$ *then* $\varphi(f)\phi = f(\lambda)\phi$.
2. $\sigma(\varphi(f)) = f(\sigma(A))$.
3. $f \geq 0 \implies \varphi(f) \geq 0$ *(this means that* $\forall \phi \in \mathcal{H}, \langle \phi \cdot \varphi(f)\phi \rangle \geq 0$.
4. $||\varphi(f)|| = ||f||_\infty$

We are going to prove the theorem and the properties altogether.

Proof. The first two items define φ uniquely on $\mathbb{C}[X]$. Since $||P(A)|| = ||P||_{\infty,\sigma(A)}$, we can extend it to $\mathcal{C}(\sigma(A), \mathbb{C})$ by density. This proves everything, but 7.1, 7.2 and 7.3.
Let's show **7.1**.

Let $f \in \mathcal{C}(\sigma(A))$, $P_n \in \mathbb{C}[X], P_n \longrightarrow f$ uniformly. Also, let $\phi, A\phi = \lambda\phi$. Then $\varphi(P_n) \longrightarrow \varphi(f)$ (because φ is an isometry). Since $\varphi(P_n)\phi = P_n(A)\phi = P_n(\lambda)\phi$, we have :

$$||\varphi(f)\phi - \varphi(P_n)\phi|| \leq ||\varphi(f) - \varphi(P_n)|| \cdot ||\phi|| = ||f - P_n||_\infty ||\phi|| \longrightarrow 0.$$

But we also have : $P_n(A)\phi = P(\lambda)\phi$. This gives $\varphi(f)\phi = f(\lambda)\phi$.
Let's prove the positivity **(7.3)**. If $f \geq 0, \varphi(f) = \varphi(\sqrt{f}) \times \varphi(\sqrt{f}) = \varphi(\overline{\sqrt{f}})\varphi(\sqrt{f}) = \varphi(\sqrt{f})^*\varphi(\sqrt{f})$. This shows that $\forall \phi \in \mathcal{H}, \langle \phi \cdot \varphi(f)\phi \rangle = \langle \phi, \varphi(\sqrt{f})^*\varphi(\sqrt{f})\phi \rangle = ||\varphi(\sqrt{f})\phi||^2 \geq 0$.
Finally, **7.2** : Let $\mu \notin \{f(\lambda), \lambda \in \sigma(A)\}$. $\{f(\lambda), \lambda \in \sigma(A)\}$ is the direct image of a compact by a continuous function, it is therefore compact. $g : \lambda \in \sigma(A) \mapsto \dfrac{1}{f(\lambda) - \mu}$ is continuous on $\sigma(A)$. $\varphi(g)(\varphi(f) - \mu I_d) = (\varphi(f) - \mu I_d)\varphi(g = \varphi(g(f - \mu)) = \varphi(1) = I_d$. This shows that $\mu \in \rho(\varphi(f))$, therefore : $\sigma(\varphi(f)) \subset f(\sigma(A))$. Finally, we have to prove that $f(\sigma(A)) \subset \sigma(\varphi(f))$. Let $\lambda \in \sigma(A)$. We suppose that $\varphi(f) - f(\lambda)I_d$ is invertible. Let $(P_n) \in \mathbb{C}[X]$ be such that $P_n \longrightarrow f$ uniformly on $\sigma(A)$. Then we have $\varphi(P_n) - P_n(\lambda)I_d \longrightarrow \varphi(f) - f(\lambda)I_d$ in $\mathcal{B}(\mathcal{H})$. Then for n large enough, $\varphi(P_n) - P_n(\lambda)I_d$ is invertible (because the set of invertible operators is open). This tells us that $P_n(\lambda) \in \sigma(\varphi(P_n)) = \sigma(P_n(A))$. But we prove earlier that $\sigma(P(A)) = P(\sigma(A))$, which tells us that $P_n(\lambda) \in P_n(\sigma(A))$, and therefore $\varphi(P_n) - P_n(\lambda)I_d$ can not be invertible.

□

2.3 Proof

Let $\phi \in \mathcal{H}$. We consider the map :

$$\begin{array}{rccc} \Psi: & \mathcal{C}(\sigma(A), \mathbb{C}) & \to & \mathbb{C} \\ & f & \mapsto & \langle \phi \cdot f(A)\phi \rangle \end{array}$$

It is linear, continuous, and non-negative.

Theorem 27 (Riesz-Markov)**.** *Let K be a compact topological space, L be a non-negative continuous linear form on $\mathcal{C}(K, \mathbb{C})$. Then, there exists a unique Borel, regular, finite measure μ on K such that:*

$$\forall f \in \mathcal{C}(K), L(f) = \int_K f d\mu.$$

Let's call μ_ϕ the function we get by applying Riesz-Markov theorem to Ψ_ϕ. A simple calculation proves that $||f(A)\phi||^2 = \int_{\sigma(A)} |f|^2 d\mu_\phi$. This proves that the map

$$\begin{array}{rccc} & \mathcal{C}(\sigma(A), ||\cdot||_{2,\mu_\phi}) & \to & \mathcal{H} \\ & f & \mapsto & f(A)\phi \end{array}$$

is an isometry.

Since our measure is regular, $\mathcal{C}(\sigma(A))$ is dense in $L^2(\sigma(A), \mu_\phi)$. L^2 being complete, we can extend the previous map in an isometry $U_\phi : L^2(\sigma(A), \mu_\phi) \longrightarrow \overline{\{f(A)\phi, f \in \mathcal{C}(\sigma(A))\}} := \mathcal{H}_\phi$.

Remark 28. *$\mathcal{H}_\phi$ is $A-$invariant.*

For all $f \in \mathcal{C}(\sigma(A)), U^*AUf = U^*Af(A)\phi = U^*(I_d \times f)(A)\phi = U^*U(I_d \times f) = I_d \times f$. (I used I_d to denote the function on $\sigma(A)x \mapsto x$...) We showed that for all $\lambda \in \sigma(A)$, $(U^*AU)(f)(\lambda) = \lambda f(\lambda) : U^*AU$ is a multiplication operator.

We got a spectral theorem on some spaces that are called "cyclic".

Property 29. *We can write $\mathcal{H} = \bigoplus_{i=1}^{N} \mathcal{H}_i$, where $N \in \mathbb{N} \cup \{\infty\}$, where $\mathcal{H}_i = \mathcal{H}_{\phi_i}$.*

Proof. I think it has axiome of choice behind it. □

Let $\mathcal{H} = \bigoplus_{i=1}^{N} \mathcal{H}_i$ be such a sum. We define a measure μ on $\sigma(A) \times \mathbb{N}^*$ by $\mu_{|\sigma(A)\times\{j\}} = \mu_{\phi_j}$ if $j \leq N$, 0 otherwise.

We thus get $U : (L^2(\sigma(A) \times \mathbb{N}^*, d\mu) \longrightarrow \mathcal{H}$ a unitary block diagonal operator such that $(U^*AU(f))(\lambda, j) = \lambda f(\lambda, j)$, this for all $f \in L^2(\sigma(A) \times \mathbb{N}^*, \mu), (\lambda, j) \in (\sigma(A), \mathbb{N}^*)$.

We have proven :

Theorem 30 (Spectral theorem). *Let $\mathcal{H}$ be a separable Hilbert space, A self-adjoint on $\mathcal{H}$. There exists a finite regular Borel measure on $\sigma(A) \times \mathbb{N}^*$ and a unitary map*

$$U : L^2(\sigma(A) \times \mathbb{N}^*, d\mu) \longrightarrow \mathcal{H}$$

such that

$$(U^* A U(f))(\lambda, j) = \lambda f(\lambda, j),$$

this for all $f \in L^2(\sigma(A) \times \mathbb{N}^*, \mu), (\lambda, j) \in (\sigma(A), \mathbb{N}^*)$.

Proof. The measure can be chosen finite because $\mathcal{H}_{\lambda_j \phi_j} = \mathcal{H}_{\phi_j}$. □

2.4 Multiplication operators

What can be said about multiplication operators ?

Property 31. *Let (X, μ) be a σ-finite measure space and let $m \in L^\infty(X, \mu)$. We define $\mathcal{H} = L^2(X, \mu)$ and $A : \mathcal{H} \longrightarrow \mathcal{H}, f \mapsto mf$. Then :*

1. $A \in \mathcal{B}(\mathcal{H})$
2. $\sigma(A) = \mathit{EssRan}(m) := \{\lambda \in \mathbb{C}, \forall \epsilon > 0, \mu(\{|m - \lambda| < \epsilon\}) > 0\}$
3. *$\lambda \in \sigma(A)$ is an eigenvalue if and only if $\mu(\{m = \lambda\}) > 0$.*

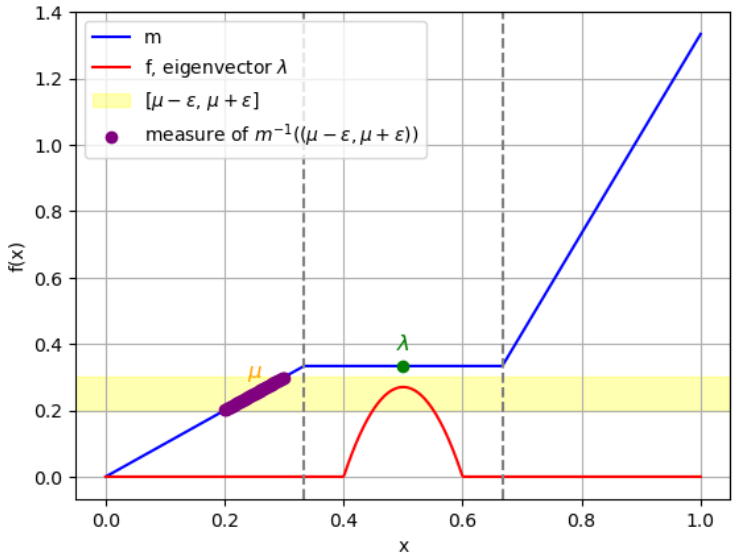


Figure 2.1: Credits : Chat-GPT. Here, μ is in $\sigma(A)$, but is not an eigenvalue. However, λ is an eigenvalue.

Proof. 1 is OK. We only prove 2 and 3. Let $\lambda \in \mathbb{C} - \textit{EssRan}(m)$. Then, there exists $\epsilon > 0$ such that $\mu(\{|m-\lambda| < \epsilon\}) = 0$. Therefore, $|m-\lambda| > \epsilon$ almost everywhere. In particuliar, the multiplication operator by $\dfrac{1}{m-\lambda}$ is a bounded operator, and an inverse of $A - \lambda I_d$. This shows that $\lambda \notin \sigma(A)$, so we have $\sigma(A) \subset \textit{EssRan}(m)$.

Let $\lambda \in$ *EssRan(A)*, $n \in \mathbb{N}^*$. Since $\mu(\left\{|m-\lambda| < \dfrac{1}{n}\right\}) > 0$, and μ is σ-finite, let $B \subset \left\{|m-\lambda| < \dfrac{1}{n}\right\}$ be such that $\mu(B) \in (0, \infty)$ (exists because the measure is $\sigma-$finite). Hence, $\mathbb{1}_B \in L^2(X, \mu)$.

We can thus define $f = \dfrac{\mathbb{1}_B}{\mu(B)^{\frac{1}{2}}}$. It is of unit norm, and $||(A-\lambda I_d)f|| = ||(m-\lambda)f||_2 \leq \dfrac{1}{n}$. We thus created a normalized sequence of almost eigenvectors. This proves that $\lambda \in \sigma(A)$.

If $\lambda \in \sigma(A)$ is an eigenvalue, there exists $f \in L^2(X, \mu)$, $||f||_2 = 1$ and $(A - \lambda I_d)f = 0$, i.e $(m(X) - \lambda)f(x) = 0$ a.e, i.e $f = 0\ a.e$ on $\{m \neq \lambda\}$. Therefore, $\mu(\{m = \lambda\}) > 0$. Conversely, if $\mu(\{m = \lambda\}) > 0$, we can construct an eigenvector like we did for 1. □

Let us apply this result to $X = \sigma(A) \times \mathbb{N}^*$, and $m(\lambda, j) = \lambda$. Since U^*AU is the multiplication operator by m, its spectrum is $\sigma(A)$. So $\mu(]\lambda - \epsilon, \lambda + \epsilon[> 0)$. In other words : μ charges every neighborhood of $(\lambda, \mathbb{N}^*)$.

And $\lambda \in \sigma(A)$ is an eigenvalue iff $\mu(\{\lambda\} \times \mathbb{N}^*) = \displaystyle\sum_{j=1}^{+\infty} \mu(\lambda, j) > 0$.

We deduce that dim ker$(A - \lambda I_d) = \#\{j \in \mathbb{N}^*, \mu((\lambda, j)) > 0\}$

Remark 32. *The $\mathbb{N}^*$ in $\sigma(A) \times \mathbb{N}^*$ plays the role of a multiplicity.*

Remark 33. *U and μ are not unique. However, dim ker$(A - \lambda I_d) = \#\{j \in \mathbb{N}^*, \mu((\lambda, j)) > 0\}$ tells us that two different μ must charge $\sigma(A) \times \mathbb{N}$ in similar ways.*

2.5 Examples

In many cases, A can be diagonalized explicitly, without using the spectral theorem.

Example 34. $\mathcal{H} = \ell^2(\mathbb{Z}, \mathbb{C})$, $A(u_n) = (u_{n+1} + u_{n-1})$.

A is bounded : if $u \in \mathcal{H}$, $||Au|| \leq 2||u||$. It is self adjoint : $\langle v \cdot Au \rangle = \displaystyle\sum_{n \in \mathbb{Z}} \overline{v_n}(u_{n+1} + u_{n-1}) = \langle Av \cdot u \rangle$ (separate the sums, reindexate, and put them back together...). Other proof : the adjoint of the left shit T is the right shift. We're considering $A = T + T^*$, which is self adjoint.

We consider :

$$\begin{array}{rccc} U: & \ell^2(\mathbb{Z}) & \to & L^2((0, 2\pi), dx) \\ & (u_n) & \mapsto & \displaystyle\sum_{n \in \mathbb{Z}} u_n e^{inx} \end{array}$$

For all $f \in L^2((0,2\pi),dx)$, $U^*(f) = (c_n(f))$ (the Fourier coefficients). Therefore, $UAU^*f = \sum_{n\in\mathbb{Z}}(c_{n+1}(f)+c_{n-1}(f))e^{inx} \underset{\textbf{(angle-moitié)}}{=} \sum_{n\in\mathbb{Z}} c_n(f)e^{inx}2\cos(x) = 2cos(x)f(x)$.

So UAU^* is a multiplication operator by $x \mapsto \cos(x)$ on $L^2((0,2\pi),dx)$. In particuliar, $\sigma(A) = [-2,2]$. However, it has no eigenvectors.

Exercise 35. *1. (a) Prove the spectral theorem for normal operators.*

(b) Let U be a normal operator on $\mathcal{H}$. Prove that U is unitary if and only if $\sigma(U) \subset \mathbb{U}$. (Hint : Use the spectral theorem for normal operators.)

2. Let $A = A^*$. Show that if $\lambda \in \rho(A)$, $||(A-\lambda I_d)^{-1}|| = \dfrac{1}{d(\lambda,\sigma(A))}$.

Proof. For exercise 2 : The direct sense is a consequence of the Neumann criterion. YOu don't need the spectral theorem for that. If $\sigma(U) \subset \mathbb{U}$, using the spectral theorem we have a unitary U such that UAU^* is a multiplication operator $f \mapsto mf$. Since $\sigma(\mathbb{U}) \subset \mathbb{U}$, Essran$(m) \subset \mathbb{U}$. But the adjoint of $f \mapsto mf$ is $f \mapsto \overline{m}f$. Since we have Essran$(m) \subset \mathbb{U}, m\overline{m} = 1$. This proves that UAU^* is unitary, and therefore A is unitary. □

Proof. For exercise 3 : We actually only need A to be normal. If A is normal, then $(A-\lambda I_d)^{-1}$ is normal. Therefore $||(A-\lambda I_d)^{-1}|| = \sup_{\mu\in\sigma((A-\lambda I_d)^{-1})} |\mu| = \sup_{\mu\in\sigma(A)} \dfrac{1}{|\lambda-\mu|} = \dfrac{1}{\inf_{\mu\in\sigma(A)}|\lambda-\mu|} = \dfrac{1}{d(\lambda,\sigma(A))}$. □

Chapter 3

Unbounded operators

3.1 Introduction

We begin with an example.

Example 36 (Multiplication operator)**.** *$f \mapsto mf$ in $L^2(X,\mu)$, with $m \notin L^\infty(X,\mu)$ is not continuous, and not even well-defined. For instance, on $\ell^2(\mathbb{N})$, $(Au)_n = (nu_n), \forall n \in \mathbb{N}, \forall u \in \ell^2$. For each n, $(Au)_n$ is well defined, yet (Au) might not be in $\ell^2(\mathbb{N})$.*

To avoid the previously mentioned case, we will restrict the domain of definition of our operator A.

Definition 37 (Unbounded operator)**.** *An unbounded operator (or just operator) on $\mathcal{H}$ is a couple $(D(A), A)$ where $D(A)$ (for domain of the operator) is a dense subspace of $\mathcal{H}$ and A is a linear map $D(A) \longrightarrow \mathcal{H}$.*

Remark 38. *We can also work with subspaces that aren't dense, however, it is more comfortable and it is the most common situation.*

Example 39. *$\mathcal{H} = \ell^2(\mathbb{N})$, $D(A_1) = \left\{u \in \ell^2(\mathbb{N}), \exists \mathbb{N} \in \mathbb{N}, \forall n \in \mathbb{N}, u_n = 0\right\}$. $D(A_1)$ is dense in ℓ^2, and for all $u \in D(A_1), (nu_n) \in \ell^2(\mathbb{N})$. Therefore, $(D(A), A_1)$ is an operator, where $A_1 : u \mapsto (nu_n)$.*

Example 40. *$\mathcal{H} = \ell^2(\mathbb{N})$, $D(A_2) = \left\{u \in \ell^2, (nu_n) \in \ell^2\right\}$ .. It is also dense in $\ell^2(\mathbb{N})$, therefore, $(D(A_2), A_2)$ is an operator, where $A_2(u) = (nu_n) \forall u \in \ell^2$.*

Example 41. *$\mathcal{H} = L^2(\mathbb{R})$, $D(A_3) = C_c^\infty(\mathbb{R})$, $A_3(f) = f', f \in D(A_3)$. $D(A_3)$ is dense in $\mathcal{H}$, therefore $(D(A_3), A_3)$ is an operator.*

Definition 42 (Extension of an operator)**.** *Let $(D(A), A)$ and $(D(B), B)$ be two operators on $\mathcal{H}$. We say that $(D(B), B)$ is an extension of $(D(A), A)$ if $D(A) \subset D(B)$, and $A = B$ on $D(A)$.*

Example 43. *A_2 is an extension of A_1.*

Remark 44. *We denote B is an extension of A by $A \subset B$. This makes sense because of the next definition.*

Definition 45 (Graph of an operator). *We define* $G(A) = \{(u, Au), u \in D(A)\} \subset \mathcal{H} \times \mathcal{H}$. *Then,* $A \subset B \iff G(A) \subset G(B)$.

Definition 46 (Spectrum of an operator). $\lambda \in \mathbb{C}$ *belongs to the resolvent set* $\rho(A)$ *of* $(D(A), A)$ *if* $A - \lambda I_d : D(A) \longrightarrow \mathcal{H}$ *is invertible, and* $(A - \lambda I_d)^{-1} : \mathcal{H} \longrightarrow D(A)$ *is bounded. The spectrum is* $\sigma(A) = \mathbb{C} - \rho(A)$.

Example 47. $\sigma(A_1) = \mathbb{C}$ *because* $\forall \lambda \in \mathbb{C}, u \in D(A_1), (A_1 - \lambda I_d)u \in D(A)$ *: not surjective. While* $\sigma(A_2) = \mathbb{N}$. *Indeed, if* $k \in \mathbb{N}$, *with* $e_k = (\delta_{k,n})_n$, $A_2 e_k = k e_k$. *Then* $k \in \sigma(A_2)$. *And if* $\lambda \in \mathbb{C} - \mathbb{N}$, $(\frac{1}{n-\lambda}) \in l^\infty(\mathbb{N})$, *same with* $(\frac{n}{n-\lambda})$. *We thus consider* $B : u \in \ell^2 \mapsto \big(\frac{u_n}{n-\lambda}\big)$. *Then* $B(A_2 - \lambda I_d) = I_{d_{D(A_2)}}, (A_2 - \lambda I_d)B = I_{d_{\mathcal{H}}}$. *And* B *is bounded, so* $\lambda \in \rho(A_2)$.

Definition 48. *An operator* A *on* $\mathcal{H}$ *is closed if its graph* $G(A)$ *is a close subset of* $\mathcal{H} \times \mathcal{H}$.

Remark 49. *If* $D(A) = \mathcal{H}$, A *is closed if and only if* A *is continuous (closed graph theorem).*

Remark 50. A *is closed* $\iff \forall (u_n) \in D(A)^{\mathbb{N}}, \forall (u, v) \in \mathcal{H} \times \mathcal{H}$, *we have :*

$$\big(\lim_{n \longrightarrow +\infty} u_n = u, \lim_{n \longrightarrow +\infty} Au_n = v\big) \implies (u \in D(A), v = Au).$$

Example 51. A_1 *is not closed because* $u_n^{(k)} = \dfrac{1}{(n+1)^2} \mathbb{1}_{n \leq k}$ *verifies :* $\lim_{k \longrightarrow +\infty} u^{(k)} = u$ *in* $\ell^2(\mathbb{N})$, *and* $\lim_{k \longrightarrow +\infty} A_1 u^{(k)} = v$. *Yet,* $u \notin D(A_1)$.

However :

Example 52. A_2 *is a closed operator. Indeed, if* $u^{(k)}$ *is a sequence in* $D(A_2)$ *satisfying* $\lim_{k \longrightarrow +\infty} u^{(k)} = u$, $\lim_{k \longrightarrow +\infty} A_2 u^{(k)} = v$, *both convergences happening in* $\ell^2(\mathbb{N})$, *then, we have, for all* $n \in \mathbb{N}$, *term by term convergences :* $\forall n \in \mathbb{N}$, $\lim_{k \longrightarrow +\infty} u_n^{(k)} = u_n$ *and* $\lim_{k \longrightarrow +\infty} u_n^{(k)} = v_n$

But $\lim_{k \longrightarrow +\infty} A_2 u_n^{(k)} = \lim_{k \longrightarrow +\infty} n u_n^{(k)} = n u_n = v_n$. *In particuliar,* $n u_n = v_n \forall n \in \mathbb{N}$. *Therefore,* $(n u_n) \in \ell^2(\mathbb{N})$, *i.e* $u \in D(A_2)$ *and* $A_2 u = v$.

Actually, only closed operators are interesting.

Property 53. *If* A *is not closed, then* $\sigma(A) = \mathbb{C}$.

Proof. If $\sigma(A) \neq \mathbb{C}$, then $\rho(A) \neq \emptyset$. Let's show that A is closed. Let $\lambda \in \rho(A)$. Therefore $(A - \lambda I_d)$ is invertible, and its inverse is bounded. Let $(u_n) \in D(A)^{\mathbb{N}}$, with $\lim_{n \longrightarrow +\infty} u_n = u$, $\lim_{n \longrightarrow +\infty} Au_n = v$ where both convergences are in $\mathcal{H}$. Since $(A - \lambda I_d)^{-1} : \mathcal{H} \longrightarrow D(A)$ is bounded, $(A - \lambda I_d)^{-1} A u_n \longrightarrow (A - \lambda Id)^{-1} v$

And writing $Au_n = (A - \lambda I_d)u_n + \lambda u_n$, we get : $(A - \lambda I_d)^{-1} A u_n = u_n + \lambda (A - \lambda I_d)^{-1} u_n \longrightarrow u + \lambda (A - \lambda I_d)^{-1} u$. Therefore, $u = (A - \lambda I_d)^{-1}(v - \lambda u) \in D(A)$. And we also have : $Au = A(A - \lambda I_d)^{-1}(v - \lambda u) = v - \lambda u + \lambda (A - \lambda I_d)^{-1}(v - \lambda u) = v$. (we once again used the trick $A = A - \lambda I_d + \lambda I_d$.) This proves that A is closed. □

Property 54. *$\rho(A)$ is open.*

Proof. Same proof as for bounded operators.

□

Remark 55. *The spectrum can be unbounded, which was not the case with bounded operators. It can also be empty, which was also not the case with bounded operators.*

Example 56. *Let $\mathcal{H} = L^2((0,1))$, $D(A) = \{f \in \mathcal{H}, f' \in \mathcal{H}, f(0) = 0\}$ (the derivative here is a distributional derivative).*
The condition "$f(0) = 0$" can be seen as weird since our function is in L^2, and defined a.e. However, we know that since its derivative is in L^2, it has a continuous representative ($\int_0^x f'(y)dy$).
$D(A)$ is dense because it contains $C_c^\infty((0,1))$.
We will thus consider $A : f \mapsto f'$. Let us show that $\sigma(A) = \emptyset$.
Let $\lambda \in \mathbb{C}$, and let us show that $A - \lambda I_d$ is invertible. Let $g \in \mathcal{H}, f \in D(A)$. $(A - \lambda I_d)f = g \iff f' - \lambda f = g$ in $\mathcal{D}'((0,1)) \iff (e^{-\lambda x}f)' = e^{-\lambda x}g \iff e^{-\lambda x}f = c + \int_0^x e^{\lambda y}g(y)dy$ in $D'((0,1))$ and $C^0((0,1))$. This gives us $f = Ce^{\lambda x} + \int_0^x e^{\lambda(x-y)}g(y)dy \forall x \in (0,1)$. Since $f(0) = 0$, we have to take $c = 0$, which makes us define :
$(B(g))(x) = \int_0^x e^{\lambda(x-y)}g(y)dy$ for all $g \in \mathcal{H}, x \in (0,1)$. Then $Bg \in C^0([0,1]) \subset \mathcal{H}$. $Bg(0) = 0$, and $(Bg)' = \lambda Bg + g$. This tells us that $Bg \in D(A)$ and $(A - \lambda I_d)B = I_{d_\mathcal{H}}$. So B is a right inverse for $(A - \lambda I_d)$. It remains to show that it is also a left inverse, and continuous. Let $f \in D(A)$. $B(A - \lambda I_d)f = \int_0^x e^{\lambda(x-y)}(f'(y) - \lambda f(y))dy \overset{IPP}{=} f(x)$, and this $\forall x \in [0,1]$.
The integration by parts is done for H^1 functions, which works because we have a continuous representative.

Finally, let's prove that B is continuous. Let $g \in \mathcal{H}$, $|Bg(x)| \leq \int_0^x |e^{\lambda(x-y)}||g(y)|dy \leq \int_0^1 |e^{\lambda(x-y)}||g(y)|dy = \int_0^1 e^{Re(\lambda)(x-y)}|g(y)dy \leq e^{Re(\lambda)} \int_0^1 |g(y)|dy \overset{CYS}{\leq} e^{Re(\lambda)}||g||_2$ Therefore, $||Bg||_2 \leq ||Bg||_\infty \leq e^{Re(\lambda)}||g||_2$, and B is bounded.

Remark 57. *$\lambda \in \sigma(A) \iff (A - \lambda I_d)$ is not injective, or not surjective, or bijective but its inverse is not continuous). There is an extra thing to check, which we did not have for bounded operator.*

However, this saves the day:

Exercise 58. *Assume that A is closed, and that $(A - \lambda I_d)$ is bijective. Show that $(A - \lambda I_d)^{-1}$ is continuous. Hint : Endow $D(A)$ with he norm $||u|| + ||Au||$.*

Proof. We remark that $D(A)$ is complete for the norm $||u|| + ||Au||$ if and only if A is closed. Since A is closed, $D(A)$ is complete and we can use the closed graph theorem. We just have to use that $(A - \lambda I_d)$ is bijective, the closed graph theorem for $(A - \lambda I_d)^{-1}$ works. □

3.2 Closable operators

Definition 59 (Closable operator)**.** *A is closable if it has a closed extension*

Property 60. *A is closable if and only if the closure of its graph in $\mathcal{H} \times \mathcal{H}$ is a graph, which happens if and only if we have : $(\forall v \in \mathcal{H}, (0, v) \in \overline{G(A)} \iff v = 0)$.*

Proof. First, we assume A is closable. Let B a closed extension of A. This means that $G(A) \subset G(B)$, and $G(B)$ is closed. Therefore, $\overline{G(A)} \subset G(B)$. Let $v \in \mathcal{H}$ be such that $(0, v) \in \overline{G(A)}$. Therefore, $(0, v) \in G(B)$, and $v = B(0) = 0$.

Now, we assume that $(\forall v \in \mathcal{H}, (0, v) \in \overline{G(A)} \iff v = 0)$. We now construct an operator B that's a closure of B. We define : $D(B) = \left\{u \in \mathcal{H}, \exists v \in \mathcal{H}, (u, v) \in \overline{G(A)}\right\}$. Let $u \in D(B)$, let $v, v' \in \mathcal{H}$ be such that $(u, v), (u, v')$ belong to $\overline{G(A)}$. Therefore, since $\overline{G(A)}$ is a vector space, $(0, v - v') \in \overline{G(A)}$. Therefore, $v = v'$. We showed : $\forall u \in D(B), \exists! v \in \mathcal{H}, (u, v) \in \overline{G(A)}$. For $u \in D(B)$, we define $Bu = v$. It is well defined by uniqueness, and is linear (easy to check). Since $G(A) \subset \overline{G(A)}$, $D(A) \subset D(B)$ and $\forall u \in D(A), Bu = Au$. It is closed because its graph is closed : $G(B) = \overline{G(A)}$. □

Definition 61. *The B constructed above is called the closure of A, denoted by $\overline{A}$. It is the smallest closed extension of A : if $A \subset B$, where B is closed, then we have $G(A) \subset \overline{G(A)} = G(\overline{A}) \subset G(B)$.*

Example 62 (A non closable operator)**.** $\mathcal{H} = L^2(]0, 1[), D(A) = C^0([0, 1])$. *We define* $A : D(A) \longrightarrow \mathcal{H}, f \mapsto \left(Af : x \mapsto f(\frac{1}{2})\right)$.

It is not closable : let $\lambda \in \mathbb{C}$, and $f_n = \lambda \times n(\frac{1}{n} - |x - \frac{1}{2}|)_+$. We have $f_n \longrightarrow 0 \in \mathcal{H}$, and $Af_n \longrightarrow \lambda \in \mathcal{H}$. Therefore, $(0, \lambda) \in \overline{G(A)} \forall \lambda \in \mathbb{C}$, and A is not closable.

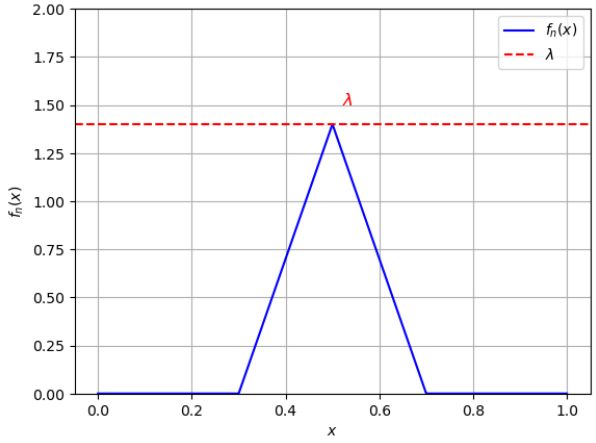


Figure 3.1: f_n

3.3 Adjoints

Definition 63. *Let A be an operator, we define $D(A^*) = \{u \in \mathcal{H}, \exists C > 0, \forall v \in D(A), |\langle u \cdot Av\rangle| \leq C||v||\}$ $= \{u \in \mathcal{H}, v \in D(A) \mapsto \langle u \cdot Av\rangle$ is continuous$\}$. Then :*

1. *For any $u \in D(A^*)$, there exists a unique element of $\mathcal{H}$, denoted by A^*u, such that $\forall v \in D(A), \langle u \cdot Av\rangle = \langle A^*u, v\rangle$.*
2. *$u \in D(A^*) \mapsto A^*u$ is linear.*
3. *$(D(A^*), A^*)$ is closed, and is called the adjoint of A.*
4. *A is closable if and only if $D(A^*)$ is dense.*

Proof. **1 and 2** follow from Riesz Lemma, and density of $D(A)$.
For 3 : Let us show that $G(A^*) = \{(-Av, v), v \in D(A)\}^\perp$. Let $u \in D(A^*)$. Then $\forall v \in D(A), \langle u \cdot Av\rangle = \langle A^*u, v\rangle$. Therefore, $\langle (u, A^*u), (-Av, v)\rangle_{\mathcal{H}\times\mathcal{H}} = 0$. Now, let $(u, w) \in \mathcal{H}\times\mathcal{H}$, such that $\forall v \in D(A), \langle (u, w) \cdot (-Av, v)\rangle = 0$. Therefore, $\forall v \in D(A), |\langle u, Av\rangle| = |\langle w, v\rangle| \leq ||w|| \cdot ||v||$. Therefore, $u \in D(A^*)$ and $A^*u = w$.
For 4 : We prove that $D(A^*)^\perp = \{0\}$. Let $v \in D(A^*)^\perp$. Let us show that $(0, v) \in \overline{G(A)}$. Since A is closable, this indeed implies $v = 0$. Let $(u, w) \in G(A)^\perp$. Let us show $\langle (0, v) \cdot (u, w)\rangle = 0$ (because this is equal to $\langle v, w\rangle$). Let us show that $w \in D(A^*)$. Since $(u, w) \in G(A)^\perp$, $\forall v \in D(A), \langle (u, w) \cdot (v, Av)\rangle = 0$. Therefore, $\langle w \cdot Av\rangle = -\langle u, v\rangle$. Therefore, $|\langle w, Av\rangle| \leq ||u|| \cdot ||v||$ and $w \in D(A^*)$. □

Remark 64. *If A is bounded, $D(A^*) = \mathcal{H}$ and A^* is bounded since $\langle u, Av\rangle \leq ||A|| \cdot ||u|| \cdot ||v||$.*

Remark 65. $A \subset B \implies B^* \subset A^*$.

Remark 66. *Even if your operator is not closed, if it has an adjoint, it will be closed.*

Property 67. *If A is closable, then $(A^*)^* = \overline{A}$ and $\left(\left(A^*\right)^*\right)^* = \overline{A}^* = A^*$.*

Proof. Denote by $R : \mathcal{H} \times \mathcal{H} \longrightarrow \mathcal{H} \times \mathcal{H}$ the rotation $R(x, y) = (-y, x)$. We prove that $G(A^*) = R\big(G(A)\big)^\perp$. We notice that $(RF)^\perp = R(F^\perp)$. Then $G((A^*)^*) = R(G(A^*))^\perp = R((R(G(A))^\perp)^\perp = (R(R(G(A))))^{\perp\perp} = G(A)^{\perp\perp} = \overline{G(A)} = G(\overline{A})$ □

Property 68. *Let A be closed. Then we have : ker$(A^* - \overline{\lambda} I_d) = \big(Ran(A - \lambda I_d)\big)^\perp = \big(Ran(\overline{A} - \lambda I_d)\big)^\perp$.*

Proof. $u \in \ker(A^* - \overline{\lambda} I_d) \iff (u, \overline{\lambda} u) \in G(A^*) \iff \forall v \in D(A), -\langle Av, u\rangle + \langle v, \overline{\lambda} u\rangle = 0 \iff \forall v \in D(A), \langle (A - \lambda I_d)v, u\rangle = 0 \iff u \in (\text{Ran}(A - \lambda I_d))^\perp$. The last equality follows from the first one and $A^* = \overline{A}^*$. □

Example 69 (Explicit computation of an adjoint)**.** *Let $\mathcal{H} = L^2(]0, 1[)$, and let :*
$D(A) = \left\{f \in \mathcal{H}, f' \in L^2(]0, 1[), f(0) = 0\right\}$ *and A the operator such that $Af = f'$. Let $g \in D(A^*)(\subset \mathcal{H})$. Then, $\forall f \in D(A), \langle g, Af\rangle = \langle A^*g, f\rangle$, which translates as : $\displaystyle\int_0^1 \overline{g(x)} f'(x)dx =$*

$\int_0^1 \overline{A^*g(x)}f(x)dx$. *In particular, for* $\varphi \in C_c^\infty(]0,1[) \subset D(A)$. *We have, with the distribution bracket :* $\langle \overline{g}, \varphi' \rangle_{\mathcal{D}'} = \langle \overline{A^*g}, \varphi \rangle$. *Therefore,* $\overline{g}' = -\overline{A}^* g$ *in* $\mathcal{D}'(]0,1[)$. *In particular,* $g' \in L^2(]0,1[)$ *and* $\forall g \in D(A^*), A^*g = -g'$. *Now,* $\forall f \in D(A)$, *we can integrate by parts (our functions are in* H^1 *where we happen to have a continuous representative)*

$$\int_0^1 \overline{g}(x)f'(x)dx = \overline{g}(1)f(1) - \overline{g}(0)f(0) - \int_0^1 \overline{g'}(x)f(x)dx.$$

This can be rewritten as : $\langle g \cdot Af \rangle = \langle A^*g \cdot f \rangle = \int_0^1 \overline{g}(x)f'(x)dx = \overline{g}(1)f(1) + \langle -g' \cdot f \rangle = \int_0^1 \overline{g}(x)f'(x)dx = \overline{g}(1)f(1) - \overline{g}(0)f(0) + \langle A^*g \cdot f \rangle$. *But if* $f \in D(A)$, *so* $f(0) = 0$. *Therefore :* $\langle A^*g \cdot f \rangle = \overline{g}(1)f(1) + \langle A^*g \cdot f \rangle$. *This gives us* $f(1) = 0$ *(we just have to choose a convenient g). Therefore,* $D(A^*) \subset \left\{ g \in L^2, g' \in L^2, g(1) = 0 \right\}$. *Now, we prove the converse. Let* $g \in L^2$, *with* $g' \in L^2$ *and* $g(1) = 0$. *Let* $f \in D(A)$. $\langle g \cdot Af \rangle = \int_0^1 \overline{g(x)}f'(x)dx \overset{IBP}{=} -\int_0^1 \overline{g'(x)}f(x)dx$. *This tells us :* $|\langle g, Af \rangle| \leq ||g'||_2 \cdot ||f||_2$. *So we have* $g \in D(A^*)$.

Exercise 70. *Replace* $D(A)$ *by* $\left\{ f \in L^2, f' \in L^2, f(\frac{1}{2}) = 0 \right\}$.

3.4 Symmetry

Definition 71 (Symmetric operator)**.** *An operator* A *is symmetric if :* $\forall u, v \in D(A), \langle u \cdot Av \rangle = \langle Au \cdot v \rangle$.

Remark 72. *In particular,* $D(A) \subset D(A^*)$, *and* $\forall u, v \in D(A), Au = A^*u$. *In other words, the adjoint is an extension of the operator. And since we know that* A^* *is closed, this tells us that all symmetric operator are closable!*

Remark 73. *A symmetric operator has two notable extensions :* $\overline{A}$, *and* A^*. *These two can differ.*

Remark 74. *Since* $\overline{A}^* = A^*$, *we have* $G(\overline{A}) = \overline{G(A)} \subset G(A^*) = G(\overline{A}^*)$. *This tells us that* $\overline{A}$ *is also symmetric.* A^* *is symmetric* $\iff G(A^*) \subset G(A^{**}) = G(\overline{A}) \subset G(A^*) \iff A^* = \overline{A}$.

Remark 75. *If* B *is a symmetric extension of* A, *then* $G(A) \subset G(B) \subset G(B^*) \subset G(A^*)$. *If* B *is also closed,* $G(\overline{A}) \subset G(B) \subset G(A^*)$. *This tells us that* $G(A^*)$ *is the largest closed extension.*

Property 76. *Let* A *be a symmetric operator. Then, we have one of the following :*

1. $\sigma(A) = \mathbb{C}$.
2. $\sigma(A) = \{z \in \mathbb{C}, Im(z) \geq 0\}$,
3. $\sigma(A) = \{z \in \mathbb{C}, Im(z) \leq 0\}$,
4. $\sigma(A) \subset \mathbb{R}$, *and* $\sigma(A) \neq \emptyset$.

Moreover, if $\lambda \in \mathbb{C} - \mathbb{R}$, then $A - \lambda I_d$ is always injective, and if it is surjective then A is closed and $\lambda \in \rho(A)$. In that case, $\sigma(A)$ is included in the half set that does not contains λ.

Remark 77. *If A is bounded, then $\sigma(A) \subset \mathbb{R}$.*

Proof. Since A is symmetric, if $u \in D(A)$, we have $||(A - \lambda I_d)u||^2 = ||(A - Re(\lambda)I_d)u||^2 + Im(\lambda)^2||u||^2$ (we did that already for bounded operators...). In particular, if $\lambda \in \mathbb{C} - \mathbb{R}$, then $A - \lambda I_d$ is injective. (sum of positive terms..) And if $A - \lambda I_d$ is bijjective, we have an estimation of the norm of the inverse operator : $||(A - \lambda I_d)^{-1}|| \leq \dfrac{1}{|I_m(\lambda)|}$ (apply the formula to a well chosen (u). This shows that if $\lambda \in \mathbb{C} - \mathbb{R} \cap \rho(A)$, then $B(\lambda, |Im(\lambda)|) \subset \rho(A)$.

We suppose that $\rho(A) \cap \mathbb{C}_+ := \{z \in \mathbb{C}, Im(z) > 0\} \neq \emptyset$. Then $\mathbb{C}_+ \subset \rho(A)$. Indeed, since $\rho(A) \cup \mathbb{C}_+$ is open in $\mathbb{C}_+$. However, it is also closed. For that, we take a squence $(\lambda_n) \in \mathbb{C}_+$, that is convergent to $\lambda \in \mathbb{C}_+$. Then $Im(\lambda_n) \longrightarrow \lambda > 0$. Let n such that $|\lambda_n - \lambda| \leq \dfrac{Im(\lambda)}{2}$ and $Im(\lambda_n) \geq \dfrac{Im(\lambda)}{2}$. Then we have $|\lambda_n - \lambda| \leq Im(\lambda_n)$, and we showed that $\lambda \in B(\lambda_n, Im(\lambda_n)) \subset \rho(A)$. By connectedness, $\rho(A) \cap \mathbb{C}_+ = \mathbb{C}_+$. This allows us to prove the first part of the property, treating the different cases. It only remains to prove that in the case $\sigma(A) \subset \mathbb{R}$, then $\sigma(A) \neq \emptyset$, which will be done later. We now assume that $A - \lambda I_d$ is surjective (therefore bijective), where $Im(\lambda) \neq 0$. Then, by the relation $||(A - \lambda I_d)u||^2 = ||(A - Re(\lambda)I_d)u||^2 + Im(\lambda)^2||u||^2$, we have $||(A - \lambda I_d)^{-1})v|| \leq \dfrac{1}{|Im(\lambda)|}||v||$. Therefore, $(A - \lambda I_d)^{-1}$ is bounded. Since $\lambda \in \rho(A)$, $\sigma(A) \neq \mathbb{C}$, and A is closed.

Now, we assume $\sigma(A) = \emptyset$. In particular, $0 \in \rho(A)$, and A is invertible. If $\lambda \neq 0 \in \rho(A)$, then $\lambda^{-1} \in \rho(A)$. Therefore, $\sigma(A^{-1}) \subset \{0\}$. But A^{-1} is a bounded self adjoint operator (write it down!). But since $||A^{-1}|| = \sup\limits_{\mu \in \sigma(A^{-1})} |\mu| = 0$, contradiction. □

Example 78. $D(A_1) = \left\{u \in \ell^2, \exists N \in \mathbb{N}, u_n = 0 \forall n \geq N\right\}$, $A_1(u) = (nu_n)$. *We've seen that $\sigma(A_1) = \mathbb{C}$. A_1 is symmetric.*

Example 79. A_2 *: just replace $D(A_1)$ by $D(A_2) = \left\{u \in \ell^2, \sum n^2|u_n|^2 < \infty\right\}$. We've seen that $\sigma(A_2) = \mathbb{N} \subset \mathbb{R}$. It is symmetric too.*

Example 80. $D(A_3) = \left\{f \in L^2(]0, +\infty[, f' \in L^2(]0, +\infty[, f(0) = 0\right\} = H_0^1(]0, +\infty[)$, *and $A_3 f = -if'$. It is symmetric, and $\sigma(A) = \overline{\mathbb{C}_-}$.*

Proof. By integration by parts, and using $f(0) = 0$: $\langle g \cdot Af\rangle = \displaystyle\int_0^{+\infty} \overline{g(x)}(-if(x))dx = i\int_0^{+\infty} \overline{g'(x)}f(x)dx = \langle Ag \cdot f\rangle$. Now, we compute $\sigma(A)$. We could try and realize A as a multiplication operator, and use the spectral theorem, but there is no obvious multiplication operator form. Let $\lambda \in \mathbb{C}, f \in D(A), g \in \mathcal{H} = L^2$. $(A - \lambda I)f = g \iff -if' - \lambda f = g \in \mathcal{D}'(]\prime, +\infty[)$. We can solve this, and we get $(A - \lambda I)f = g \iff \forall x \in [0, 1], f(x) = i\displaystyle\int_0^x e^{i\lambda(x-y)}g(y)dy$. (we use $f(0) = 0$ here too). Now, we want to see when does this function belongs to $D(A_3)$. If f is in L^2, because of the equation, f' is in L^2 too. So we have

to check for which $\lambda : x \mapsto i\int_0^x e^{i\lambda(x-y)}g(y)dy := Bg(x)$ is in L^2. If $Im(\lambda) > 0$, then Bg is L^2. Indeed, if we do $y' = x - y$, we get $Bg(x) = \int_0^x e^{i\lambda y'}g(x-y')dy$. Therefore, $|Bg(x)| \leq \int_0^{+\infty} e^{-Im(\lambda)y'}g(x-y')dy'$. This is a convolution $L^2 - L^1$, which is L^2. Therefore, $\{Im(\lambda) > 0\} \subset \rho(A)$, and $\sigma(A) \subset \overline{\mathbb{C}_-}$. If $Im(\lambda) = 0, g(y) = \frac{e^{i\lambda y}}{1+y}$ is in L^2, and $Bg(x) = ie^{i\lambda x}\int_0^x \frac{1}{1+y}dy = ie^{i\lambda x}\ln(1+\lambda) \notin L^2$. Therefore, $\lambda \in \sigma(A)$ and $\mathbb{R} \subset \sigma(A)$. If $Im(\lambda) < 0$, then with $g(y) = e^{-y}$ we have $g \in L^2$ and $Bg(x) = ie^{i\lambda x}\int_0^x e^{-(1+i\lambda)y}dy = i\frac{e^{i\lambda x} - e^{-x}}{1+i\lambda}$. This is not in L^2. We have thus proven that $\{Im(\lambda) < 0\} \subset \sigma(A)$, and we then have $\sigma(A) = \{Im(z) \leq 0\}$. □

3.5 Self-adjointness

Definition 81. *A is self-adjoint if $A^* = A$, i.e :*

1. *A is symmetric*
2. *$D(A^*) \subset D(A)$*

Theorem 82. *Let A be a symmetric operator. The following are equivalent :*

1. *A is self-adjoint*
2. *$\sigma(A) \subset \mathbb{R}$*
3. *$\exists \lambda \in \mathbb{C}, (A - \lambda I_d)$ and $(A - \overline{\lambda} I_d)$ are both surjective.*

Proof. $1 \implies 2$: If $\lambda \in \mathbb{C} - \mathbb{R}$. If $(A - \lambda I_d)$ is surjective, then $\lambda \in \rho(A)$. $\text{Ran}(A - \lambda I_d)^\perp = \text{Ran}(A^* - \lambda I_d)^\perp = \ker(A - \overline{\lambda} I_d) = \{0\}$. So the range is dense, and closed because we have $||(A - \lambda I_d)u|| \geq C||u||$ for all $u \in D(A)$. This shows that $\lambda \in \rho(A)$.
$2 \implies 3$: Since $\sigma(A) \subset \mathbb{R}$, $\pm i \in \rho(A)$. So $A \pm iI_d$ are surjective.
$3 \implies 1$: Let us show that $D(A^*) \subset D(A)$. Let $u \in D(A^*)$. Then : $\langle v \cdot Au\rangle = \langle A^* v \cdot u\rangle$. This gives us : $\langle v \cdot (A - \lambda I_d)u\rangle = \langle (A^* - \overline{\lambda} I_d)v \cdot u\rangle$. Thanks to our hypothesis, we can write $(A - \overline{\lambda} I_d)\tilde{v} = (A^* - \overline{\lambda} I_d)v$. Then $\langle (A^* - \overline{\lambda} I_d)v \cdot u\rangle = \langle (A - \overline{\lambda} I_d)\tilde{v} \cdot u\rangle = \langle \tilde{v} \cdot (A - \lambda I_d)u\rangle$.
Therefore : $\langle v \cdot (A - \lambda I_d)u\rangle = \langle \tilde{v} \cdot (A - \lambda I_d)u\rangle$. Then we have $\langle v - \tilde{v} \cdot (A - \lambda I_d)u\rangle = 0 \forall u \in D(A)$
But since $(A - \lambda I_d)$ is surjective, we get $v = \tilde{v}$. □

Definition 83 (Essentially self-adjoint operator)**.** *If $\overline{A}$ is self-adjoint, we say that A is essentially self-adjoint.*

Theorem 84. *Let $A = A^*$, $\lambda \in \mathbb{R}$. Then, the following are equivalent :*

1. $\lambda \in \sigma(A)$,
2. $\inf_{u \in D(A), ||u||=1} ||(A - \lambda I_d)u|| = 0$,
3. $\exists (u_n) \in D(A)^{\mathbb{N}}, ||u_n|| = 1, ||(A - \lambda I_d)u_n|| \longrightarrow 0$.

Proof. 2 and 3 are clearly equivalent. If 2 is not true, then there exists $c > 0$ such that $||(A - \lambda I_d)u|| \geq c||u||$ for all $u \in D(A)$. Then $(A - \lambda I_d)$ is injective, has closed range and if it is bijective its inverse is bounded. But we have seen that $Ran(A - \lambda I_d)$ is dense when $A = A^*$. So $\lambda \in \rho(A)$. 3 implies 1 : we have done that a few weeks ago. □

Example 85. *$\mathcal{H} = L^2(\mathbb{R})$. $D(A) = H^1(\mathbb{R})$, $A(f) = -if'$ for $f \in D(A)$. It is symmetric by integration by parts. It is self adjoint : We need to find a λ such that $A - \lambda I_d$ and $A - \overline{\lambda} I_d$ are surjective. For instance, $\lambda = i$. Let $g \in L^2(\mathbb{R})$. Let's show that $g \in Ran(A + iI_d)$. Indeed, $(A+iI_d)f = g$ if and only if $-if'+if = g$ in $\mathcal{D}'(\mathbb{R})$, but also in $\mathcal{S}'(\mathbb{R})$. This is interesting because we can use a Fourier Transform there. We use the convention : $\hat{\varphi}(\zeta) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \varphi(x) e^{-ix\zeta} dx$. This gives us : $-\hat{if'} + i\hat{f} = \hat{g}$ in $\mathcal{S}'(\mathbb{R})$. Which gives us : $(\zeta + i)\hat{f} = \hat{g}$ in $\mathcal{S}'(\mathbb{R})$. Define $Bg \in \mathcal{S}'(\mathbb{R})$ by $\hat{Bg} = \frac{1}{\zeta + i}\hat{g} \in L^2$ ($L^\infty \times L^2$). But we also have $\zeta \mapsto \zeta \hat{Bg}(\zeta) \in L^2$, which gives us $(Bg)' \in L^2$. Then $Bg \in D(A)$, and $||Bg||_2 \leq C||g||_2$. Furthermore, for all $f \in D(A), B(A + Id)f = f$ and $(A + iI_d)Bg = g$. Then, $A + iId$ is invertible. The same argument proves that $A - I_d$ is invertible. (it would have been enough to prove that it is surjective, but the more the merrier). We thus proved that A is self adjoint, using the last theorem. Let us show that $\sigma(A) = \mathbb{R}$.*

Let $\varphi \in C_c^\infty(\mathbb{R})$, $||\varphi||_2 = 1$. We define $\psi_n \in D(A)^{\mathbb{N}}$ by $\hat{\psi_n}(\zeta) = \sqrt{n}\hat{\varphi}(n(\zeta - \lambda))$. The $\sqrt{n}$ ensured that it is normalized. The support of $\hat{\psi_n}$ is concentrated around λ.

$$||(A - \lambda I_d)\psi_n||^2 \overset{\textbf{Fourier is an isometry}}{=} \int_{\mathbb{R}} |\zeta - \lambda|^2 |\hat{\psi_n}(\zeta)|^2 d\zeta = \frac{1}{n^2} \int_{\mathbb{R}} |\zeta|^2 |\hat{\varphi}(\zeta)|^2 d\zeta \longrightarrow 0.$$

This concludes.

Exercise 86. *$D(A) = \left\{ f \in \mathcal{H}, f'' \in L^2(\mathbb{R}) \right\}$, $Af = f''$. Show that A is self-adjoint, and determine $\sigma(A)$.*

Chapter 4

Concrete examples

4.1 Derivation

Here, $\mathcal{H} = L^2(]0,1[), D(A) = C_c^\infty(]0,1[)$, and $Af = -if'$. A is symmetric (integration by parts).

4.1.1 Closure of A

Let $(f,g) \in \overline{G(A)}$, $(f_n) \subset D(A)^{\mathbb{N}}$ such that $f_n \longrightarrow f$ and $-if_n' \longrightarrow g$, both convergences being in L^2. We also have convergence in $\mathcal{D}'(]0,1[)$. But $f_n' \longrightarrow f'$ in $\mathcal{D}'(\mathbb{R})$. Therefore, $f' = ig \in L^2$. Conclusion : $(f,g) = (f, -if'), f \in H^1(]0,1[)$. However, it is not true that $H_1 \subset \overline{C_c^\infty}$. (The closure happens to be H_0^1).

So with $D = H_0^1$, we have :

1. $\forall (f,g) \in \overline{G(A)}, f \in D$ and $g = -if'$
2. $\forall f \in D, \exists (f_n) \subset D(A), f_n \longrightarrow f$ and $f_n' \longrightarrow f'$, both convergences in $L^2 : (f, -if') \in \overline{G(A)}$.

Therefore, $\overline{G(A)} = G(\overline{A})$ where $D(\overline{A}) = D$ and $\forall f \in D, \overline{A}f = -if'$. We already know that its closure is symmetric, but one can check it by integration by parts

4.1.2 Adjoint of A

Let $g \in D(A^*)$. For all $f \in C_c^\infty(]0,1[)$, $\langle A\overline{A^*g} \cdot f\rangle_{\mathcal{D}',\mathcal{D}} = \displaystyle\int_0^1 \overline{A^*g(x)} f(x)dx = \int_0^1 \overline{g(x)}(-if'(x))dx = \langle (i\overline{g})' \cdot f\rangle_{\mathcal{D}',\mathcal{D}}$. Therefore, $(-ig)' = \underbrace{A^*g}_{\in L^2(]0,1[)}$ in $\mathcal{D}'$. This tells us that $g \in H^1$, and $A^*g = -ig'$.

$|\langle g \cdot Af\rangle| = \Big| \displaystyle\int_0^1 \overline{g(x)}(-if'(x))dx\Big| \overset{IBP+f(0)=f(1)=0}{=} \Big| \int_0^1 \overline{-ig'(x)} f(x)dx\Big| \overset{CYS}{\leq} ||g'||_2||f||_2$.

In conclusion : $D(A^*) = H^1$. In particular, $D(A) \not\subset D(A^*)$ and $D(\overline{A}) \not\subset D(A^*)$. This way : A is not self adjoint, nor its closure : it is not essentially self adjoint. Moreover, A^* is not symmetric, but A is : the domain causes a lot of problems.

4.1.3 Self adjoint extensions ?

Let $B = B^*$ an extension of A. Since it is self adjoint, it is closed. Therefore, $\overline{A} \subset B$. Moreover, since $A \subset B$, then $\underbrace{B^*}_{=B} \subset A^*$. In particular, $D(B) \subset D(A^*) = H^1$, and for all $f \in D(B)$, $Bf = A^*f = -if'$. Since B is symmetric,

$$\forall f, g \in D(B), 0 = \langle g \cdot Bf\rangle - \langle Bg \cdot f\rangle \overset{IBP}{=} -i(f(1)\overline{g(1)} - f(0)\overline{g(0)}). \tag{4.1}$$

Since $\overline{A} \subset B, D(\overline{A}) = H_0^1 \subset D(B)$. Since $\overline{A} \neq B$ because $B = B^*$, $\overline{A} \neq \overline{A}^* = A^*$, we know that $D(\overline{A}) \neq D(B)$. Let $f_0 \in D(B)$, such that $f_0 \notin H_0^1$. Hence, we either have $f_0(0) \neq 0$ or $f_0(1) \neq 0$. With 4.1, with $f = g = f_0$, we get $|f_0(1)|^2 = |f_0(0)|^2 \neq 0$. We thus get $\theta \in \mathbb{R}$, such that $f_0(1) = e^{i\theta} f_0(0)$.

But with 4.1, with $f = f_0, g \in D(B)$, we get $e^{i\theta} f_0(0)\overline{g(1)} = f_0(0)\overline{g(0)}$.Therefore, $g(1) = e^{i\theta} g(0)$, where θ does not depend on g.

We therefore define, for $\theta \in \mathbb{R}$:

$$D(B_\theta) = \left\{f \in H^1, f(1) = e^{i\theta} f(0)\right\}, \textit{and } B_\theta f = -if' \tag{4.2}$$

We have proved $D(B) \subset D(B_\theta)$. Reciprocally, if $g \in D(B_\theta)$, we have, $\forall f \in D(B)$:

$$\left|\langle g \cdot Bf\rangle\right| \overset{IBP}{=} \left| \int_0^1 \overline{-ig'(x)} f(x) dx\right| \leq ||g'||_2 ||f||_2.$$

Therefore, $g \in D(B^*) = D(B)$.

We prove that $D(B) = D(B_\theta)$. The same computation shows $B_\theta = B_\theta^*$, foral all $\theta \in \mathbb{R}$. The self-adjoint extensions are exactly $\{B_\theta, \theta \in \mathbb{R}\}$. Example : for $\theta = 0$, we get boundary conditions $f(0) = f(1)$.

4.1.4 Spectrum of B_θ

Let $\lambda \in \mathbb{C}, g \in L^2(]0,1[)$.

$$(B_\theta - \lambda I_d)f = g \iff -if' - \lambda f = g \iff \forall x \in [0,1], f(x) = Ce^{i\lambda x} + i\int_0^x e^{i\lambda(x-y)} g(y) dy.$$

Where $f(0) = C, f(1) = Ce^{i\lambda} + i\displaystyle\int_0^1 e^{i\lambda(1-y)} g(y) dy$. If we want to have $f(1) = e^{i\theta} f(0)$, it gives us : $C(e^{i\theta} - e^{i\lambda}) = i\displaystyle\int_0^1 e^{i\lambda(1-y)} g(y) dy$.

If $\lambda \in \theta + 2\pi\mathbb{Z}$, then $e^{i\theta} - e^{i\lambda} = 0$, and $g(y) = e^{i\lambda y} \in L^2$, but $g \notin \text{Ran}(B_\theta - \lambda I_d)$. Hence, $\lambda \in \sigma(B_\theta)$ and $\ker(B_\theta - \lambda I_d) = \mathbb{C}(x \mapsto e^{i\lambda x}) := \mathbb{C}f_\lambda$. $f_\lambda(1) = e^{i\lambda} = e^{i\theta} = e^{i\theta} f_\lambda(0)$.

If $\lambda \notin \theta + 2\pi\mathbb{Z}$, then $e^{i\theta} - e^{i\lambda} \neq 0$, and if we define, for $g \in L^2$:

$$Cg(x) = \frac{i}{e^{i\theta} - e^{i\lambda}} \int_0^1 e^{i\lambda(1-y)} g(y)dy + i \int_0^x e^{i\lambda(x-y)} g(y)dy. \tag{4.3}$$

We thus have :

$Cg \in D(B_\theta)$, and :

$$\begin{cases} (B_\theta - \lambda I_d)C = I_{d_{L^2}} \\ C(B_\theta - \lambda I_d) = I_{d_{D(B_\theta)}}. \end{cases}$$

This proves that $\lambda \in \rho(B_\theta)$.

Conclusion : $\sigma(B_\theta) = \theta + 2\pi\mathbb{Z}$.

Remark : for $\theta = 0$, by Fourier series, B_0 is unitarily conjugated to the multiplication by $k \mapsto 2\pi k$ on $\ell^2(\mathbb{Z})$.

Exercise 87. *Same thing with $L^2(]0, +\infty[)$.*

4.2 Laplacian on $]0, 1[$

4.2.1 First definitions

$\mathcal{H} = L^2(]0, 1[)$, $D(A) = C_c^\infty(]0, 1[)$, $Af = -f'' \forall f \in D(A)$.

1. $D(A)$ is dense in $\mathcal{H}$.
2. A is symmetric : double IBP. Use the fact that both f and its derivative vanish at the boundary.

4.2.2 Closure of A

Let $(u, v) \in \overline{G(A)}$, and (u_n) such that $\begin{cases} u_n \longrightarrow u \in \mathcal{H} \\ Au_n \longrightarrow v \in \mathcal{H}. \end{cases}$

Since $u_n \longrightarrow u \in \mathcal{H}$, $u_n \longrightarrow u \in \mathcal{D}'$, and $u_n'' \longrightarrow u'' \in \mathcal{D}'$. Hence, $u'' = -v \in L^2$. In particuliar, $u \in \{f \in \mathcal{H}, f'' \in \mathcal{H}\}$.

Lemma 88 (Lemma about Sobolev Spaces)**.** *Let $u \in \mathcal{H}$.*

1. *If $u' \in \mathcal{H}$, then $u \in C^0([0, 1])$, and :*
   - *(a)* $\forall x, y \in [0, 1], |u(x) - u(y)| \leq ||u'||_{L^2(]x,y[)} \sqrt{|x - y|}$
   - *(b)* $\forall x \in [0, 1], |u(x)| \leq C(||u||_2 + ||u'||_2)$
2. *If $u'' \in \mathcal{H}$, then $u' \in C^0([0, 1])$ and : $\forall x \in [0, 1], |u'(x)| \leq C(||u||_2 + ||u''||_2)$.*

Proof. First : $\forall x, y \in]0,1[$:

$$|u(x) - u(y)| = \Big| \int_y^x u'(t)dt \Big| \overset{CYS}{\leq} ||u'||_{L^2(]x,y[)} \sqrt{|x-y|} \leq ||u'||_2.$$

This gives us:

$$|u(x)| \leq |u(y)| + ||u'||_2.$$

Integrating this, and using Cauchy-Schwarz, we get :

$$|u(x)| \leq ||u||_2 + ||u'||_2. \tag{4.4}$$

Now, for the rest :

$$u'(x) - u'(y) = \int_y^x u''(t)dt.$$

Integrating it between $0 < t < t' < 1$:

$$(t-t')u'(x) = u(t) - u(t') + \int_t^{t'} \int_y^x u''(z)dzdy$$

Which finally gives us :

$$|t-t'||u'(x)| \leq |u(t)| + |u(t')| + ||u''||_2.$$

Integrating this, we get :

$$\Big(\underbrace{\int_0^1 \int_0^1 |t-t'|dtdt'}_{>0} \Big) ||u'(x)| \overset{CYS}{\leq} 2||u_2|| + ||u''||_2.$$

Which concludes. □

From the lemma we deduce :

$$\left.\begin{array}{l} u_n \longrightarrow u \in \mathcal{H} \\ Au_n \longrightarrow v \in \mathcal{H}. \end{array}\right\} \Longrightarrow \left\{\begin{array}{l} u_n' \longrightarrow u' \in C^0([0,1]) \\ u_n \longrightarrow u \in C^0([0,1]). \end{array}\right.$$

Since we have $u_n(0) = 0 = u_n(1) = u_n'(0) = u_n'(1)$ for all $n \in \mathbb{N}$, we deduce that :

$$u \in D = \{f \in \mathcal{H}, f'' \in \mathcal{H}, f(0) = f(1) = f'(0) = f'(1) = 0\} = H_0^2(]0,1[).$$

This way, $\overline{G}(A) \subset \big\{(u, -u''), u \in H_0^2(]0,1[).\big\}$.

Reciprocally, let $u \in D$, let us construct $(u_n) \in D(A)$, such that : $\left\{\begin{array}{l} u_n \longrightarrow u \in \mathcal{H} \\ u_n'' \longrightarrow u'' \in \mathcal{H}. \end{array}\right.$

Let χ be a function $\mathbb{R}_+ \longrightarrow \mathbb{R}$, such that $\chi = 0$ on $[0, \frac{1}{2}]$, $\chi = 1$ on $[1, +\infty]$ and χ is increasing on $[\frac{1}{2}, 1]$.

We define $\chi_\varepsilon(x) = \chi(\frac{x}{\varepsilon})$. Clearly, $\chi_\varepsilon u$ is close to $u \in \mathcal{H}$. We also have that $(\chi_\varepsilon u)''$ is close to u'' in $\mathcal{H}$. To show that, we have to make a calculation using the lemma. I do not feel like typing it in LaTeX right now, it will therefore be an exercise.

This function is close to u in L^2, same for its second derivative, and its support does not touch 0. If (φ_n) is an approximation of unity that is in $C_c^\infty(\mathbb{R})$, we can make the convolution $\chi_\varepsilon u * \phi_n$. This is $C_c^\infty(]0,1[)$ for n large enough.

Conclusion : $\overline{G(A)} = \{(u, -u''), u \in D\}$, so that $D(\overline{A}) = D = H_0^2$,

4.2.3 Adjoint of A

Same argument as for $-if'$, we get $D(A^*) = \{f \in \mathcal{H}, f'' \in \mathcal{H}\} = H^2(]0,1[)$. And for all $f \in D(A^*), A^* f = -f''$. Therefore, $A \neq A^*, \overline{A} \neq A^*$.

4.2.4 Self-adjoint extensions?

If $B = B^*, A \subset B$, then $\overline{A} \subset B \subset A^*$. In particular, $D(B) \subset D(A^*) = H^2(]0,1[)$,and $\forall f \in D(B), Bf = A^f = -f''$. Moreover,

$$\begin{aligned} 0 = \langle g\dot{B}f\rangle - \langle Bg \cdot f\rangle &= \overline{g(0)}f'(0) - \overline{g(1)}f'(1) + \overline{g'(1)}f(1) - \overline{g'(0)}f(0) \\ &= \left\langle \begin{pmatrix} g(0) \\ g'(0) \\ g(1) \\ g'(1) \end{pmatrix} \cdot \underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}}_{M} \begin{pmatrix} f(0) \\ f'(0) \\ f(1) \\ f'(1) \end{pmatrix} \right\rangle \end{aligned} \tag{4.5}$$

We define $V = \{(f(0), f'(0), f(1), f'(1)), f \in D(B)\} \subset \mathbb{C}^4$.

Then 4.5 is equivalent to $V \subset (MV)^\perp$.

Since M is invertible (rank..), we have : $\dim(V) \leq \dim(MV)^\perp = 2 - \dim(V) \leq 2$. Let us show that $B = B^*$ if and only if $\dim(V) = 2$.

Indeed, if $(a,b,c,d) \in (MV)^\perp$, then let $g \in H^2(]0,1[)$ be such that $(g(0), g'(0), g(1), g'(1)) = (a,b,c,d)$ (we can choose a polynomial for instance).

IBP tells us that $\langle g, Bf\rangle = \underbrace{\left\langle \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \cdot M \begin{pmatrix} f(0) \\ f'(0) \\ f(1) \\ f'(1) \end{pmatrix} \right\rangle}_{=0} - \int_0^1 \overline{g''(x)} f(x) dx$. This tells us that : $|\langle g \cdot Bf\rangle| \leq ||g''||_2 ||f||_2$, and $g \in D(B^*) = D(B)$. This tells us that $(g(0), g'(0), g(1), g'(1)) \in V \implies (MV)^\perp \subset V$. We thus have $V = (MV)^\perp$, and dim(V) = 2.

Reciprocally, if $V = (MV)^\perp$, let $D(A_V) = \left\{f \in H^2(]0,1[), (f(0), f'(0), f(1), f'(1)) \in V\right\}$ (notice that $D(\overline{A}) \subset D(A_V)$).

$\forall f \in D(A_V), A_V f = -f''$. But since $V \subset (MV)^\perp$, A_V is symmetric. If $g \in D(A_V^*), g \in H^2(]0,1[)$ and an integration by parts shows that $(g(0), g'(0), g(1), g'(1)) \in (MV)^\perp = V$. Therefore, $D(A_V^*) \subset D(A_V)$. This proves that A_V is self adjoint.

If B is a self adjoint extension of A, we have seen that $B \subset A_V$ where

$$V = \{(f(0), f'(0), f(1), f'(1)), f \in D(B)\}$$

, so $B \subset A_V = A_V^* \subset B^*$, and $B = B^* \implies B = A_V$.

4.2.5 Examples of V

Example 89 (Dirichlet boundary condition). *If $V = span(e_1, e_4)$, we have:*

$$D(A_v) = \left\{f \in H^2(]0,1[), f(0) = f(1) = 0\right\}$$

. This is the Dirichlet boundary conditions.

***Spectrum**: Let $\lambda \in \mathbb{R}, f \in D(A_V), g \in L^2(]0,1[)$. We want to know when we can solve $-f'' - \lambda f = g$.*

1. *If $\lambda > 0$: we get $f(x) = \cos(\sqrt{\lambda x})a + \sin(\sqrt{\lambda x})b - \int_0^x \frac{\sin(\sqrt{\lambda}(x-y))}{\sqrt{\lambda}} g(y)dy$. The boundaries conditions give : $f(0) = 0 \iff a = 0$, and the condition $f(1) = 0$ can be realized if and only if $\sqrt{\lambda} \notin \pi\mathbb{N}^*$. Therefore, if $\sqrt{\lambda} \in \pi\mathbb{N}^*, \sqrt{\lambda} \in \sigma(A_V)$ and $ker(A_v - \sqrt{\lambda} I_d) = \mathbb{C}\sin(\sqrt{\lambda}x)$.*
2. *If $\lambda = 0$, we get $f(x) = a + bx - \int_0^x (x-y)g(y)dy$. Here, $f(0) = 0 \iff a = 0$, and $f(1) = 0 \iff b = \int_0^1 (1-y)g(y)dy$. There are always solutions, and $0 \in \rho(A_V)$.*
3. *If $\lambda < 0$: $f(x) = ach(\sqrt{-\lambda x} + bsh(\sqrt{-\lambda}x) - \int_0^x \frac{sh(\sqrt{-\lambda}(x-y))}{\sqrt{-y}} g(y)dy$. Here the boundaries give us : $f(0) = 0 \iff a = 0$, $f(1) = 0 \implies bsh(\sqrt{-\lambda}) = \int_0^1 \frac{sh(\sqrt{-\lambda}(1-y)}{\sqrt{-\lambda}} g(y)dy$, which is always solvable. Therefore, $\lambda \in \rho(A_V)$.*

***Conclusion** : We prove that $\sigma(A_V) = \left\{\pi^2 k^2, k \in \mathbb{N}^*\right\}$ and $\forall k \in \mathbb{N}^*$, $ker(A_V - \pi^2 k^2 I_d) = \mathbb{C}(x \mapsto \sin(\pi k x))$.*

Example 90 (Neumann boundary conditions). *If $V = span(e_1, e_3)$, then :*

$$D(A_V) = \left\{f \in H^2(]0,1[), f'(0) = f'(1) = 0\right\}$$

.

Here, we find $\sigma(A_V) = \left\{\pi^2 k^2, k \in \mathbb{N}\right\}$, and $ker(A_V - \pi^2 k^2 I_d) = \mathbb{C}(x \mapsto \pi k x)$.

Example 91 (Periodic laplacian). *If* $V = span(e_1 + e_2, e_2 + e_4)$, *then :*

$$D(A_V) = \left\{ f \in H^2(]0,1[), f(0) = f(1), f'(0) = f'(1) \right\}$$

In that case, $\sigma(A_V) = \left\{ 4\pi^2 k^2, k \in \mathbb{N} \right\}$, *and* $\forall k \in \mathbb{N}^*$, $ker(A_V - \pi^2 k^2 I_d) = span(x \mapsto cos(2\pi kx), x \mapsto (2\pi kx))$, *and* $ker(A_V) = span(x \mapsto 1)$.

Remark 92. *On* $L^2(\mathbb{R})$, *we saw that with* $A(f) = -f'', D(A) = H^2(\mathbb{R})$, *we had* $\sigma(A) = [0, +\infty[$. *The smaller the space, the discreter the spectrum! And in that case, we had no actual eigenvalue.*

Remark 93. *In all these examples,* $\sigma(A_V) \subset \mathbb{R}_+$. *This has to do with the fact that* $\forall f \in D(A_V), \langle f \cdot A_V f \rangle = \int_0^1 \overline{f(x)}(-if''(x))dx = [\overline{f}(-f')]_0^1 + \int_0^1 |f'(x)|^2 dx$. *All our conditions made the boundary term positive, which means that our operator is positive, and implies its spectrum is a subset of* $\mathbb{R}_+$ *: we saw that on test 1. However, a change of conditions can give negative eigenvalues, as show the following.*

Example 94 (Robin boundary conditions). $f'(0) = af(0)$, $f'(1) = -af(1)$, *where* $a \in \mathbb{C}$ *is fixed. Neumann is* $a = 0$, *Dirichlet is* $a = +\infty$. *With* $f(x) = ch(\lambda(x - \frac{1}{2})) \implies -f'' = -\lambda^2 f$, *and* $\frac{f'(0)}{f(0)} = -\frac{f'(1)}{f(1)} = \lambda th(\frac{\lambda}{2})(= a)$. *For that* $a, -\lambda^2 \in \sigma(A_V)$.

Example 95 (An example in higher dimensions). *Let* $\mathcal{H} = L^2(\mathbb{R}^d, \mathbb{C})$, $D(A) = \{f \in \mathcal{H}, \Delta f \in \mathcal{H}\}$, $Af = -\Delta f = \sum_{j=1}^{d} \frac{d^2 f}{d^2 x_j}$, *where* $f \in D(A)$. *Then* A *is self-adjoint,* $\sigma(A) = [0, +\infty[$, *and it has no eigenvectors.*

Remark 96. $D(A) = H^2(\mathbb{R}^d)$. *This is NOT obvious : we only know that the sum of the second derivatives is* L^2, *we don't know anything about the second derivatives by themselves, and even less about the simple derivatives/crossed derivatives ! This can be understood using the Fourier Transform.* $f \in D(A) \iff \hat{f} \in L^2$ *and* $\zeta \mapsto |\zeta|^2 \hat{f}(\zeta) \in L^2 \iff (1 + |\zeta|^2)\hat{f} \in L^2$. *But we know that* $\partial^\alpha f \in L^2 \iff \zeta^\alpha \hat{f} \in L^2$. *For* $|\alpha| = 0 : OK$. *For* $|\alpha| = 1 : |\zeta_k| \leq \frac{1}{2}(1 + |\zeta_k|^2)$. *For* $|\alpha = 2|, |\zeta_j \zeta_k| \leq \frac{1}{2}(\zeta_j^2 + \zeta_k^2) \leq |\zeta|$. *More generally, it is true if you replace* $\mathbb{R}$ *by* $\Omega \subset \mathbb{R}^d$. *For* $]0,1[$, *we know that* $f'' \in L^2(]0,1[) \implies f \in C^0([0,1])$, *and then it works. For other* Ω, *it can be really difficult.*

Chapter 5

Spectral theorem for unbounded operators

5.1 Unbounded multiplication operators

Property 97. *Let $B \subset \mathbb{R}^d$ be a Borel subset, μ be a locally finite measure on B (this means that $\forall R > 0, \mu(\{x \in B, ||x|| < R\}) < \infty$). Let $a \in L^2_{loc}(B, \mu)$. We define M_a by :*

$$\begin{cases} D(M_a) = \left\{ f \in L^2(B, \mu), af \in L^2(B, \mu) \right\} \\ \forall v \in D(M_a), M_a f = af \end{cases}$$

Then :

1. *M_a is a closed operator.*
2. *$\sigma(M_a) = EssRan(a)$.*
3. *$\lambda \in \sigma(M_a)$ is an eigenvalue iff $\mu(\{\lambda\}) > 0$, in which case :*
   $ker(M_a - \lambda I_d) = \left\{ f \in L^2(B, \mu), f = 0 \quad \mu - \text{ a.e on } a \neq \lambda \right\}$
4. *M_a is self adjoint $\iff$ a is real valued.*

Proof. $D(M_a)$ is dense since it contains bounded functions with bounded support.

1. If we have :
$$\begin{cases} f_n \longrightarrow f \in L^2(B, \mu) \\ af_n \longrightarrow g \in L^2(B, \mu) \end{cases}$$
then $\forall R > 0, f_n \longrightarrow f$ in $L^2(B_R, \mu)$, and $af_n \longrightarrow af$ in $L^1(B_R, \mu)$. This tells us that $af = g$ on $B_R, \forall R > 0$ So $af = g$. Since $g \in L^2, af \in L^2$, and $f \in D(M_a)$. This concludes.
2. Same proof as the bounded case, adding $\forall \lambda \notin \text{EssRan}(a), \text{Ran}(M_{(a-\lambda)^{-1}}) \subset D(M_a)$.
3. Same as in the bounded case.

4. $\forall f, g \in D(M_a)$, $\langle f \cdot M_a g\rangle - \langle M_a f \cdot g\rangle = \int_B (a(x) - \overline{a(x)})\overline{f(x)}g(x)dx$. Choosing $f = \text{sgn}(Im(a))\mathbb{1}_{B_R}, g = \mathbb{1}_{B_R}$, then M_a symmetric implies $\forall R > 0, \int_{B_R} |Im(a)|d\mu = 0$. Then $Im(a) = 0$ a.e.

   This shows us that $\sigma(M_a) = \text{EssRan}(a) \subset \mathbb{R}$. Since M_a is symmetric, we deduce that M_a is self-adjoint.

□

5.2 Continuous functional calculus

We want to define functional calculus for normal operators. For bounded operator, this was the frame :

$$A \overset{||P(A)||=||P||_{\infty(\sigma(A))}}{\longrightarrow} P(A) \overset{f\in C^0(\sigma(A))}{\longrightarrow} f(A) \longrightarrow \int_{\sigma(A)} f(\lambda)d\mu_\psi$$

.

We have a problem for the first arrow, because of domains issues, and for the second one, because we lack compactness. The strategy will be to study the resolvant, which happens to be bounded.

Definition 98. *Let $X \subset \mathbb{R}$. We define $C^0_{lim}(X) = \left\{ f : X \longrightarrow \mathbb{C}, \lim_{x\longrightarrow+\infty} f(x) = \lim_{x\longrightarrow-\infty} f(x) < \infty \right\}$*

Remark 99. *If X does not contain a neighborhood of $\pm\infty$, it is only the set of continuous functions on X.*

Theorem 100 (Continuous functional calculus)**.** *Let A be a self-adjoint operator on $\mathcal{H}$. Then there exists a unique map $\phi : \begin{cases} C^0_{lim}(\sigma(A)) & \longrightarrow B(\mathcal{H}) \\ f & \longmapsto f(A) \end{cases}$ such that :*

1. *ϕ is linear, $\phi(fg) = \phi(f)\phi(g)$, $\phi(f)^* = \phi(\overline{f})$*
2. *$\phi(\tilde{1}) = I_d$*
3. *$||\phi(f)|| = ||f||_\infty$*
4. *$\phi(x \mapsto \frac{1}{x+i}) = (A + iI_d)^{-1}$.*

Lemma 101. *Let $z \in \rho(A)$. Then $\sigma((A - zI_d)^{-1}) = \left\{ \frac{1}{\lambda - z}, \lambda \in \sigma(A) \right\} \cup \begin{cases} \emptyset & \text{if } D(A) = \mathcal{H} \\ \{0\} & \text{otherwise} \end{cases}$*

Remark 102. *If $\rho(A) \neq \emptyset$, A is closed. Then, if $D(A) = \mathcal{H}$, A is bounded (closed graph).*

Proof. Let $\lambda \in \rho(A)$.
$(A - zI_d)^{-1} - (\lambda - z)^{-1} I_d = -(\lambda - z)^{-1}(A - \lambda I_d) \underbrace{(A - zI_d - (\lambda - z)I_d)}_{A - \lambda I_d}(A - zI_d)^{-1}$. This is invertible, therefore $\dfrac{1}{\lambda - z} \in \rho((A - zI_d)^{-1})$.

Reciprocally, if $\lambda \neq z$ is such that $\dfrac{1}{\lambda - z} \in \rho((A - zI_d)^{-1})$, then $A - \lambda I_d = A - zI_d - (\lambda - z)I_d = -(\lambda - z)\big((A - zI_d)^{-1} - (\lambda - z)^{-1}\big)\big(A - zI_d\big)$ is invertible and $\lambda \in \rho(A)$.

Since $\lambda \mapsto \dfrac{1}{\lambda - z}$ is bijective $\mathbb{C} - \{z\} \longrightarrow \mathbb{C} - \{0\}$, we prove that $\rho((A - zI_d)^{-1}) - \{0\} = \left\{ \dfrac{1}{\lambda - z}, \lambda \in \rho(A), \lambda \neq z \right\}$. Then we have two cases :

1. If $0 \in \rho((A - zI_d)^{-1})$, then $\sigma((A - zI_d)^{-1}) = \left\{ \dfrac{1}{\lambda - z}, \lambda \in \sigma(A) \right\}$.
2. If $0 \notin \rho((A - zI_d)^{-1}) = \left\{ \dfrac{1}{\lambda - z}, \lambda \in \sigma(A) \right\} \cup \{0\}$.

Note that if we are in the second case, then $(A - zI_d)^{-1} : \mathcal{H} \longrightarrow \mathcal{H}$ is bijective, but since its range is $D(A)$, then $D(A) = \mathcal{H}$. □

5.2.1 Functional calculus for bounded normal operators

We will use this theorem to prove the one for unbounded operators.

Theorem 103 (Continuous functional calculs for bounded normal operators)**.** *Let $A \in B(H)$ be a normal operator. Then there exists a unique $\phi : C^0(\sigma(A), \mathbb{C}) \longrightarrow B(H)$ satisfying the followings :*

1. *ϕ is linear, $\phi(fg) = \phi(f)\phi(g)$, $\phi(\tilde{1}) = I_d$, $\phi(f)^* = \phi(\overline{f})$*
2. *$||\phi(f)|| = ||f||_{\infty, \sigma(A)}$*
3. *$\phi(\lambda \mapsto \lambda) = A$*

We recall that if f is normal, $||A|| = \sup\limits_{\lambda \in \sigma(A)} |\lambda|$, and $||P(A)|| = ||P||_{\infty, \sigma(A)}$. For complex valued Stone Weierstrass theorem, we need the subalgebra to be stable by complex conjugation. If $A = A^*$, no problems, because $\sigma(A) \subset \mathbb{R}$. But if not, $\{\lambda \in \sigma(A) \mapsto P(\lambda) \in \mathbb{C}, P \in \mathbb{C}[X]\}$ is not stable by conjugation. We need to find the right subalgebra. We will take :

$$\left\{\lambda \in \sigma(A) \mapsto P(\lambda, \overline{\lambda}), P \in \mathbb{C}(z, \overline{z})\right\}.$$

One can show, and it is very hard, hence admitted :

Theorem 104. $\sigma(P(A, A^*)) = \left\{P(\lambda, \overline{\lambda}), \lambda \in \sigma(A)\right\}$

From this follows :

Property 105. *We then have* $||P(A, A^*)|| = \sup\limits_{\lambda \in \sigma(A)} |P(\lambda, \overline{\lambda})| = \sup\{|\mu|, \mu \in \sigma(P(A, A^*))\}$

5.2.2 Proof of the continuous functional calculus for unbounded operators

First step : If $D(A) = \mathcal{H}$, since $A = A^*$, in particular A is closed. By the closed graph theorem, A is bounded. Let ϕ_B be the functional calculus map for bounded operators.

Existence: We take $\phi = \phi_B$. It satisfies clearly $1, 2, 3$. For 4 : $\phi_b(x+i) = A + i$, hence $\phi_b(x+i)\phi_b(\frac{1}{x+i}) = I_d$, which proves $\phi_b(\frac{1}{x+i}) = (A+i)^{-1}$. *Uniqueness:* Given ϕ, let us show that $\phi = \phi_b$. Since $\sigma(A)$ is compact, $x \mapsto x \in C^0_{lim}(\sigma(A))$. $\phi(x+i)\phi(\frac{1}{x+i}) = I_d \implies \phi(x+i) = A + iI_d \implies \phi(x) = A$.

Step two: If $D(A) \neq \mathcal{H}$: Define $B = (A + iI_d)^{-1}$. B is a normal operator. By the lemma, $\sigma(B) = \left\{ \frac{1}{\lambda + I}, \lambda \in \sigma(A) \right\} \cup \{0\}$. Define

$$\tau : \begin{cases} C^0_{lim}(\sigma(A)) & \longrightarrow & C^0(\sigma(B)) \\ f & \longmapsto & \begin{cases} \tau(f)(\frac{1}{\lambda+i}) = f(\lambda) & \text{if } \lambda \in \sigma(A) \\ \tau(f)(0) = \lim f \end{cases} \end{cases}$$

It is then easy to check that τ is linear, $\tau(\overline{f}) = \overline{\tau(f)}, \tau(\tilde{1}) = \tilde{1}$, that τ is bijective since $\tau^{-1}(g)(\lambda) = g(\frac{1}{\lambda+i}) \forall \lambda \in \sigma(A)$, and $||\tau(f)||_{\infty,\sigma(B)} = ||f||_{\infty,\sigma(A)}$.

Uniqueness of ϕ : If ϕ is a functional calculus map for A then $\phi \circ \tau^{-1}$ is a functional calculus for B. Hence, $\phi \circ \tau_{-1} = \phi_B$, which gives $\phi = \phi_B \circ \tau$.

Existence Define $\phi = \phi_B \circ \tau$. It is easy to check that it works.

□

Remark 106. *We have for* $\lambda \in \rho(A)$: $\phi(\frac{1}{x-\lambda}) = (A - \lambda I_d)^{-1}$, *because* $\phi(\frac{1}{x-\lambda}) = \phi(\frac{1}{x+i})\phi(\frac{x+i}{x-\lambda})$. *Thus, even though we defined functional calculus for one resolvant, we can define it for all resolvant.*

5.3 Spectral theorem for unbounded operators

Theorem 107. *Assume that* $\mathcal{H}$ *is separable, and* $A = A^*$. *Then there exists a finite Borel measure* μ *on* $\sigma(A) \times \mathbb{N} \subset \mathbb{R}^2$, *and a unitary* $U : L^2(\sigma(A) \times \mathbb{N}, \mu) \longrightarrow \mathcal{H}$ *such that if* $D = \left\{ f \in L^2((\sigma(A) \times \mathbb{N}), \mu) \int_{\sigma(A)\times\mathbb{N}} |\lambda|^2 |f(\lambda, n)|^2 d\mu < \infty \right\}$, *we have :*

1. $UD = D(A)$
2. $\forall f \in D, \forall (\lambda, n) \in \sigma(A) \times \mathbb{N}, U^*AUf(\lambda, n) = \lambda f(\lambda, n)$.

Proof. As in the proof of the bounded case, for any $\psi \in \mathcal{H}$, let $\mu_{\mathcal{H}}$ be a finite regular Borel measure on $\sigma(A)$ such that $\forall f \in C^0_{lim}(\sigma(A)), \langle \psi, f(A)\psi \rangle = \int_{\sigma(A)} f(\lambda) d\mu_\psi(\lambda)$. The map $f \in C^0_{lim}(\sigma(A)) \mapsto f(A)\psi$ can be extended to a unitary

$$U_\psi : L^2(\sigma(A), d\mu_\psi) \longrightarrow \mathcal{H}_\psi = \underbrace{\overline{\{f(A)\psi, f \in C^0_{lim}(\sigma(A))\}}}_{(A+i)^{-1}\text{invariant}}$$

such that :

$$\forall f \in L^2(\sigma(A), \mu_\psi), \forall \lambda \in \sigma(A), (U_\psi^*(A+i)^{-1}U_\psi f)(\lambda) = \frac{1}{\lambda + i}$$

Writing $\mathcal{H} = \bigoplus_j^\perp \mathcal{H}_{\psi_j}$, we build $U : L^2(\sigma(A) \times \mathbb{N}, \mu) \longrightarrow \mathcal{H}$ unitary such that :

$$\forall f \in L^2(\sigma(A) \times \mathbb{N}, \mu), \forall (\lambda, n) \in \sigma(A) \times \mathbb{N}, (U^*(A+i)^{-1}Uf)(\lambda, n) = \frac{1}{\lambda + i} f(\lambda, n).$$

Let us show that : $UD = D(A)$.

First inclusion : let $f \in D$. Then $g = (\lambda + i)f \in L^2$. Hence $Uf = \underbrace{U\frac{1}{\lambda + i}}_{=\underbrace{(A+iI_d)^{-1}}_{\mathcal{H} \longrightarrow D(A)} U} g$

Reverse inclusion : Let $u \in D(A)$. Let $v \in \mathcal{H}$ with $u = (A + iI_d)^{-1}v = \underbrace{(A+iI_d)^{-1}U}_{=U\frac{1}{\lambda+i}} U^*v =$

Uf, with $f = \frac{1}{\lambda + i}\underbrace{U^*v}_{\in L^2}$.

Define $B = M_\lambda$ on $L^2(\sigma(A) \times \mathbb{N}, \mu)$, with $D(B) = D$. We have $U(B + iI_d)\underbrace{U^*(A+iI_d)^{-1}}_{\frac{1}{\lambda+i}U^*} =$

$UU^* = I_d$. Hence, $\forall f \in D(A), U(B+iI_d)U^*f = \underbrace{U(B+iI_d)U^*(A+iI_d)^{-1}}_{=I_d}(A+iI_d)f$

So $UBU^* = A$ □

5.4 Borel Functional Calculus

Theorem 108. *Let $\mathcal{H}$ be a separable Hilbert Space, $A = A^*$. Then, there exists a unique map $\phi : \underbrace{L^\infty(\sigma(A), \mathbb{C})}_{\text{bounded Borel functions}} \longrightarrow B(\mathcal{H})$, such that :*

1. *ϕ is a $*$-algebra morphism*

2. $||\phi(f)|| \leq \sup_{\lambda \in \sigma(A)} |f(\lambda)|$

3. $\phi(\frac{1}{x+i}) = (A + iI_d)^{-1}$

4. *If* $|f_n(x)| \leq C$ *and* $\forall \lambda \in \sigma(A), f_n(\lambda) \longrightarrow f(\lambda)$.

Then, $\forall \psi \in \mathcal{H}, \phi(f_n)\psi \longrightarrow \phi(f)\psi \in \mathcal{H}$

Remark 109. *The last consequence does not mean* $\phi(f_n) \longrightarrow \phi(f)$ *in* $B(\mathcal{H})$.

Proof. **Existence:** Let (μ, U) from the spectral theorem. Define $\phi(f) = UM_{f(\lambda)}U^*$. Since f is bounded, M_f is bounded on $L^2(\sigma(A) \times \mathbb{N}, \mu) : ||Mf|| = ||f||_{\infty, \sigma(A)\times\mathbb{N}, \mu} \leq \sup_{\lambda \in \sigma(A)} |f(\lambda)|$. (iv) follows from dominated convergence :

$$||\phi(f_n) - \phi(f)\psi||^2 = \int_{\sigma(A)\times\mathbb{N}} |f_n(\lambda) - f(\lambda)|^2 |(U^*\psi)(\lambda, n)|^2 d\mu \longrightarrow 0.$$

Uniqueness : Let $\phi, \tilde{\phi}$ be two functional calculus maps on L^∞. Then, $\phi_{|C^0_{\text{lim}}} = \tilde{\phi}_{|C^0_{\text{lim}}}$ by uniqueness of the C^0_{lim} functional calculus map. Let $U \subset \sigma(A)$ be a bounded open set, then $\phi(\mathbb{1}_U) = \tilde{\phi}(\mathbb{1}_B)$ (approach it with functions in C^0_{lim}) Then, $\mathcal{F} = \left\{ B, \textit{Borel subset of } \sigma(A), \phi(\mathbb{1}_B) = \tilde{\phi}(\mathbb{1}_B) \right\}$ is a σ-algebra, and it contains all open bounded sets, hence contains all Borel set. Hence, we've shown the results for characteristics functions, and we can thus prove it for all $L^\infty(\sigma(A), \mathbb{C})$ functions, and it concludes. □

Property 110. *As a corollary : if* $A = A^*$ *on a separable* $\mathcal{H}$*, for* $f : \sigma(A) \longrightarrow \mathbb{C}$ *a Borel measurable function, locally bounded, we define* $f(A)$ *by :*

$$\begin{cases} D(f(A)) = & UD_{M_f} \\ f(A) = & UM_fU^* \end{cases}$$

Then, $f(A)$ *does not depend on the choice of* (U, μ).

Remark 111. *By the spectral theorem, for* $f(\lambda) = \lambda$*, we get* $f(A) = A$.

Proof. Let $f_k = f\mathbb{1}_{|f|\leq k}$. $f_k \in L^\infty(\sigma(A), \mathbb{C})$. Then :

$$u \in D(f(A)) \iff \int_{\sigma(A)\times\mathbb{N}^*} |f(\lambda)|^2 |(U^*f)(\lambda, n)|^2 d\mu < \int_{\sigma(A)\times\mathbb{N}^*} \infty \iff |f_k(\lambda)^2 |(U^*f)(\lambda, n)|^2 d\mu < \infty$$

$$\iff \lim_{k\longrightarrow+\infty} ||\phi(f_k)u||^2 < \infty.$$

This property does not depend on (μ, U) because ϕ does not. Next, if $u \in D(f(A))$, then

$$U^*f(A)u = f(\lambda)(U^*u)(\lambda, n) \overset{\textit{dominated convergence}}{=} L^2 - \lim_{k\longrightarrow+\infty} f_k(\lambda)(U^*u(\lambda, n)),$$

which does not depend on (μ, U) since ϕ does not. □

Remark 112. *One can extend the* (U, μ) *authorized in the theorem as any* (U, μ) *such that* $\exists a : X \longrightarrow \mathbb{R}$*,* $a \in L^2_{loc}(X, \mathbb{R})$*,* $U^*AU = M_a$*,* $UD_{M_a} = D(A)$*. That means that* a *does not necessarily have to come from the spectral theorem.*

5.5 Applications

5.5.1 Spectral projections

Definition 113. *Let $A = A^*$, $F \subset \mathbb{R}$ a Borel subset. The spectral projection of A on F is $\mathbb{1}_F(A) \in B(\mathcal{H})$. By functional calculus, $\mathbb{1}_F(A)^2 = \mathbb{1}_F(A) = \mathbb{1}_F(A)^*$. We also define $\mathbb{1}_F(A)\mathcal{H}$ to be the spectral subspace of A associated to F.*

Property 114. *1. If F is bounded, $\mathbb{1}_F(A)\mathcal{H} \subset D(A)$,*

2. $\mathbb{1}_{\{\lambda\}} = 0$ if $ker(A - \lambda I_d) = \{0\}$, and $\mathbb{1}_{\{\lambda\}} =$ orthogonal projection on $ker(A - \lambda I_d)$ otherwise.

Definition 115. *For all $\lambda \in \mathbb{R}$, we define $P_\lambda := \mathbb{1}_{]-\infty,\lambda]}(A)$. For $\psi \in \mathcal{H}, \lambda \mapsto \langle \psi \cdot P_\lambda \psi \rangle \geq 0$ is non decreasing, and bounded. Therefore, its distributional derivative is a measure, denoted by $d\langle \psi \cdot P_\lambda \psi \rangle$. One can show that $\langle \psi \cdot A\psi \rangle = \int_{\sigma(A)} \lambda d\langle \psi \cdot P_\lambda \psi \rangle$. The idea being that : $A = \int_{\sigma(A)} \lambda dP_\lambda$.*

5.5.2 Powers

If A is unbounded, A^2 is naturally defined on $D(A^2) = \{u \in D(A), Au \in D(A)\}$. Is it dense ? Do we have $A = A^* \implies A^2 = (A^2)^*$?

Property 116. *Let $A = A^*$, $f(\lambda) = \lambda^2$. Then, we have :*

1. $D(f(A)) = \{u \in D(A), Au \in D(A)\}$
2. $\forall u \in D(f(A)), f(A)u = A(Au)$.

Note that $f(A)$ is self adjoint because f is real valued.

Proof. $D(f(A)) = \left\{u \in \mathcal{H}, \lambda^2 U^* u(\lambda) \in L^2\right\} = \left\{ \underbrace{u \in \mathcal{H}, \lambda(U^* u) \in L^2}_{u \in D(A)}, \underbrace{\lambda(\lambda U^* u)(\lambda) \in L^2}_{Au \in D(A)} \right\}$. □

Property 117. *Let $A = A^*$ such that $\sigma(A) \subset [0, +\infty[$. Let $f(\lambda) = \sqrt{\lambda}$. The self adjoint operator $f(A)$ satisfies $D(A) = \{u \in D(f(A)), f(A)u \in D(f(A))\}$, and $\forall u \in D(A), Au = f(A)(f(A)u)$.*

Proof. Same as before. □

5.5.3 Solving the Scrödinger's equation

Motivation :

$$\begin{cases} iu'(t) = & Au(t),\ \textit{A self adjoint} \\ u_{|t=0} = & u_0 \end{cases}$$

(Think of $A = -\Delta, i\partial_u = -\delta u$, $u \in L^2(\mathbb{R}^d)$.) One want to say : $u(t) = e^{-itA}u_0$. But the serie representation $\sum_{n\geq 0} \frac{(-itA)^n}{n!}$ is a nightmare : we have so much domains to compute. Instead, we use functional calculus.

Theorem 118. *Let $A = A^*$, $u_0 \in \mathcal{H}$. Then there exists a unique $u \in C^0(\mathbb{R}, \mathcal{H})$ such that $u(0) = 0$, and $\forall v \in D(A), i\frac{d}{dt}\langle v \cdot u(t)\rangle = \langle Av \cdot u(t)\rangle \in D'(\mathbb{R})$.*
Furthermore, $\forall t \in \mathbb{R}, ||u(t)|| = ||u_0||$, and if $u_0 \in D(A)$, then $\forall t \in \mathbb{R}, u(t) \in D(A)$ and $u \in C^1(\mathbb{R}, \mathcal{H})$ with $\forall t \in \mathbb{R}, iu'(t) = Au(t)$.

Proof. **Existence :** For $t \in \mathbb{R}, f_t(\lambda) := e^{-it\lambda}$. $f_t \in L^\infty(\mathbb{R}, \mathbb{C})$. We define $u(t) = f_t(A)u_0$. Then, $(U^*u(t))(\lambda, n) = e^{-it\lambda}U^*u_0(\lambda, n) \in C^0(\mathbb{R}, L^2(\sigma(A)\times\mathbb{N}, \mu))$. Therefore, $u \in C^0(\mathbb{R}, \mathcal{H})$. Since $|e^{-it\lambda}| = 1, ||U^*u(t)||_2 = ||U^*u_0||_2, ||u(t)||_2 = ||u_0||$ because U is unitary.

Let $v \in D(A)$. $\langle v\cdot u(t)\rangle = \int_{\sigma(A)\times\mathbb{N}^*} \overline{(U^*v)(\lambda, n)}e^{-it\lambda}(U^*u_0)(\lambda, n)d\mu$. We see this as a function in t, which we want to differentiate. For that, we use differentiation under the integral, and the fact that $\underbrace{|\lambda||(U^*v)(\lambda, n)|}_{\in L^2_{\lambda,n} \text{ because } v\in D(A)}\underbrace{|(U^*u_0)(\lambda, n)|}_{\in L^2_{\lambda,n}} \in L^1_{\lambda,n}$. Therefore, we can differentiate under the integral and $i\frac{d}{dt}\langle v \cdot u(t)\rangle = \langle Av, u(t)\rangle$ (just write it down).

Finally, if $u_0 \in D(A)$, then $\lambda(U^*u_0)(\lambda, n) \in L^2_{\lambda,n}$, then $\lambda\underbrace{e^{-it\lambda}(U^*u_0)}_{=U^*u(t)}(\lambda, n) \in L^2(\lambda, n) \iff$ $u(t) \in D(A), \forall t \in \mathbb{R}$.

Also, $||i\frac{u(t+h)-u(t)}{h} - Au(t)||^2 = \int_{\sigma(A)\times\mathbb{N}^*} \underbrace{|i\frac{e^{-i(t+h)\lambda} - e^{-it\lambda}}{h} - \lambda e^{-it\lambda}|^2}_{=|i\frac{e^{-ih\lambda}-1}{h}-\lambda|^2 \underset{h\to 0}{\longrightarrow} 0, \textit{ but also} \leq (2|\lambda|)^2} |U^*u_0(\lambda, n)|^2 d\mu.$

But $|\lambda|^2|U^*u_0(\lambda, n)|^2 \in L^1$, so dominated convergence shows that $||i\frac{u(t+h)-u(t)}{h} - Au(t)||^2 \underset{h\to 0}{\longrightarrow} 0$, and $u \in C^1(\mathbb{R}, \mathcal{H})$ (it remains to prove the continuity of the derivative....), with $\forall t \in \mathbb{R}, iu'(t) = Au(t)$.

Uniqueness : Let $u \in C^0(\mathbb{R}, \mathcal{H}), v_0 \in D(A)$. Define $v(t) = e^{-itA}v_0$. Then $v \in C^1(\mathbb{R}, \mathcal{H})$, $\forall t \in \mathbb{R}, v(t) \in D(A), iv'(t) = Av(t)$. The map $f : (t, t') \in \mathbb{R}^2 \mapsto \langle v(t') \cdot u(t)\rangle$ is continuous, and differentiable when t' is fixed, and $\partial_t f(t, t') = -i\langle Av(t'), u(t)\rangle \in C^0_{t,t'}$ $\partial_{t'} f(t, t') = \langle -iAv(t') \cdot u(t)\rangle = i\langle Av(t') \cdot u(t)\rangle \in C^0_{t,t'}$

Therefore, $f \in C^1_{t,t'}$, and $t \mapsto f(t, t) \in C^1_t$, and $\forall t \in \mathbb{R}, \langle v(t) \cdot u(t)\rangle = \langle v_0 \cdot u_0\rangle = \langle e^{-itA}v_0 \cdot u(t)\rangle = \langle v_0 \cdot (e^{-itA})^*u(t)\rangle = \langle v_0 \cdot e^{itA}u(t)\rangle$.

Since $D(A)$ is dense, $\forall t \in \mathbb{R}, e^{itA}u(t) = u_0$, so $u(t) = e^{-itA}u_0$ (functional calculus argument).

□

Remark 119. *The functions $U(t) = e^{itA}$ verify $U(t) = e^{itA}, U(t)^* = U(-t), U(t+s) = u(t)U(s), U(0) = I_d, U(t)\psi \underset{t\longrightarrow 0}{\longrightarrow} \psi$. They form a semi-group.*

Example 120. $A = -\Delta$, $\mathcal{H} = L^2(\mathbb{R}^d)$. $\mathcal{F}(e^{it\Delta}u)(\zeta) = e^{-it|\zeta|^2}(\mathcal{F}u)(\zeta)$.